Stationary Gamma Processes

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1 Introduction

For fixed $\alpha, \beta > 0$ these notes present six different stationary time series, each with Gamma $X_t \sim \mathsf{Ga}(\alpha, \beta)$ univariate marginal distributions and autocorrelation function $\rho^{|s-t|}$ for $X_s,\ X_t$. Each will be defined on some time index set $\mathcal{T}$, either $\mathcal{T} = \mathbb{Z}$ or $\mathcal{T} = \mathbb{R}$.

Five of the six constructions can be applied to other Infinitely Divisible (ID) distributions as well, both continuous ones (normal, α-stable, *etc.*) and discrete (Poisson, negative binomial, *etc.*). For specifically the Poisson and Gaussian distributions, all but one of them (the Markov change-point construction) coincide— essentially, there is just one "$\mathsf{AR}(1)$-like" Gaussian process (namely, the $\mathsf{AR}(1)$ process in discrete time, or the Ornstein-Uhlenbeck process in continuous time), and there is just one $\mathsf{AR}(1)$-like Poisson process. For other ID distributions, however, and in particular for the Gamma, each of these constructions yields a process with the same univariate marginal distributions and the same autocorrelation but with different joint distributions at three or more times.

First, by "$\mathsf{Ga}(\alpha, \beta)$" we mean the Gamma distribution with mean α/β and variance α/β^2, *i.e.*, with shape parameter α and *rate* parameter β. The pdf and chf are given by:

$$\begin{aligned} f(x \mid \alpha, \beta) &= \frac{\beta^\alpha}{\Gamma(\alpha)}\ x^{\alpha-1} e^{-\beta x} \mathbf{1}_{\{x>0\}} \\ \chi(\omega \mid \alpha, \beta) &= (1 - i\omega/\beta)^{-\alpha} \\ &= \exp\left\{ - \int_{\mathbb{R}_+} e^{i\omega u}\ \alpha e^{-\beta u}\, u^{-1}\, du \right\}. \end{aligned} \tag{1}$$

The sum $\xi_+ := \sum \xi_j$ of independent random variables $\xi_j \overset{\text{ind}}{\sim} \mathsf{Ga}(\alpha_j, \beta)$ with the same rate parameter also has a Gamma distribution, $\xi_+ \sim \mathsf{Ga}(\alpha_+, \beta)$, if $\{\alpha_j\} \subset \mathbb{R}_+$ are summable with sum $\alpha_+ := \Sigma \alpha_j < \infty$. Eqn (1) shows that the "Lévy measure" for this distribution is $\nu(du) = \alpha e^{-\beta u}\, u^{-1} \mathbf{1}_{\{u>0\}} du$.

From the Poisson representation of ID distributions

$$X = \int_{\mathbb{R}} u \mathcal{N}(du)$$

for $\mathcal{N}(du) \sim \mathsf{Po}\big(\nu(du)\big)$ we can show that the extremal properties of ID random variables depend only on the Lévy measure $\nu(du)$: for large $u \in \mathbb{R}_+$,

$$\begin{aligned} \mathsf{P}[X > u] &\approx \mathsf{P}\big[\mathcal{N}\big((u,\infty)\big) \neq \emptyset\big] \\ &= 1 - \exp\big(-\nu\big((u,\infty)\big)\big) \\ &\approx \nu\big((u,\infty)\big) \approx (\alpha/\beta u)e^{-\beta u}, \end{aligned} \tag{2}$$

from which one might mount a study of the multivariate extreme properties of the six processes presented below.

2 Six Stationary Gamma Processes

2.1 The Gamma $\mathsf{AR}(1)$ Process

Fix $\alpha, \beta > 0$ and $0 \le \rho < 1$. Let $X_0 \sim \mathsf{Ga}(\alpha, \beta)$ and for $t \in \mathbb{N}$ define X_t recursively by

$$X_t := \rho X_{t-1} + \zeta_t \tag{3}$$

for iid $\{\zeta_t\}$ with chf $\mathsf{E}e^{i\omega\zeta_t} = (1 - i\omega/\beta)^{-\alpha}(1 - i\rho\omega/\beta)^{\alpha} = \big[\frac{\beta - i\omega}{\beta - i\rho\omega}\big]^{-\alpha}$ (easily seen to be positive-definite, with Lévy measure $\nu_\zeta(du) = \alpha\left[e^{-\beta u} - e^{-\beta u/\rho}\right] u^{-1}\mathbf{1}_{\{u>0\}}du$). A simple way of generating $\{\zeta_t\}$ with this distribution is presented in the Appendix. The process $\{X_t\}$ has Gamma univariate marginal distribution $X_t \sim \mathsf{Ga}(\alpha, \beta)$ for every $t \in \mathbb{R}_+$ and, at consecutive times $0, 1$, joint chf

$$\begin{aligned} \chi(s,t) &= \mathsf{E}\exp(isX_0 + itX_1) \\ &= \mathsf{E}\exp(i(s+\rho t)X_0 + it\zeta_1) \\ &= \Big[\frac{(1 - i(s+\rho t)/\beta)\ (1 - it/\beta)}{1 - it\rho/\beta}\Big]^{-\alpha}. \end{aligned} \tag{4}$$

Since this is asymmetric in s, t, the process is not time-reversible; this is also evident from the observation that

$$\mathsf{P}[X_t \ge \rho X_{t-1}] = \mathsf{P}[\zeta_t \ge 0] = 1 > \mathsf{P}[\zeta_t \le X_{t-1}(1-\rho^2)/\rho] = \mathsf{P}[X_{t-1} \ge \rho X_t].$$

This process has marginal distribution $X_t \sim \mathsf{Ga}(\alpha, \beta)$ at all times t, and has autocorrelation $\mathsf{Corr}\big(X_s, X_t\big) = \rho^{|s-t|}$ (easily found from either (3) or (4)). It is clearly Markov (from (3)), and can be shown to have infinitely-divisible (ID) multivariate marginal distributions of all orders, since the $\{\zeta_t\}$ are ID.

2.2 Thinned Gamma Process

If $X \sim \mathsf{Ga}(\alpha_1, \beta)$ and $Y \sim \mathsf{Ga}(\alpha_2, \beta)$ are independent, then $Z := X + Y \sim \mathsf{Ga}(\alpha_+, \beta)$ and $U := X/Z \sim \mathsf{Be}(\alpha_1, \alpha_2)$ are also independent (where $\alpha_+ := \alpha_1 + \alpha_2$), because under the change of variables $x = uz$, $y = (1-u)z$ with Jacobian $J(u, z) = z$,

$$
\begin{aligned}
f(u, z) &= \left\{ \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} x^{\alpha_1 - 1} e^{-\beta x} \right\} \left\{ \frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} y^{\alpha_2 - 1} e^{-\beta y} \right\} J(u, z) \\
&= \left\{ \frac{\beta^{\alpha_+}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \right\} u^{\alpha_1 - 1} (1-u)^{\alpha_2 - 1} z^{\alpha_+ - 2} e^{-\beta z} z \\
&= \left\{ \frac{\Gamma(\alpha_+)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} u^{\alpha_1 - 1} (1-u)^{\alpha_2 - 1} \right\} \quad \left\{ \frac{\beta^{\alpha_+}}{\Gamma(\alpha_+)} z^{\alpha_+ - 1} e^{-\beta z} \right\}.
\end{aligned}
$$

Thus if $Z \sim \mathsf{Ga}(\alpha_1 + \alpha_2, \beta)$ and $U \sim \mathsf{Be}(\alpha_1, \alpha_2)$ are independent then $X = UZ \sim \mathsf{Ga}(\alpha_1, \beta)$ and $Y = (1-U)Z \sim \mathsf{Ga}(\alpha_2, \beta)$ are independent too. Let

$$X_0 \sim \mathsf{Ga}(\alpha, \beta)$$

and, for $t \in \mathbb{N}$ (or $t \in -\mathbb{N}$, resp.) set

$$X_t := \xi_t + \zeta_t$$

where

$$
\begin{aligned}
&\xi_t := B_t \cdot X_{t-1}, \qquad B_t \sim \mathsf{Be}(\alpha\rho, \alpha\bar{\rho}), \\
&\zeta_t \sim \mathsf{Ga}(\alpha\bar{\rho}, \beta)
\end{aligned}
$$

(or $\xi_t = B_t \cdot X_{t+1}$, resp.), where $\bar{\rho} := (1-\rho)$ and all the $\{B_t\}$ and $\{\zeta_t\}$ are independent.

Then, by induction, $\xi_t \sim \mathsf{Ga}(\alpha\rho, \beta)$ and $\zeta_t \sim \mathsf{Ga}(\alpha\bar{\rho}, \beta)$ are independent, with sum $X_t \sim \mathsf{Ga}(\alpha, \beta)$. Thus $\{X_t\}$ is a Markov process with Gamma univariate marginal distribution $X_t \sim \mathsf{Ga}(\alpha, \beta)$, now with symmetric joint chf

$$
\begin{aligned}
\chi(s, t) &= \mathsf{E} \exp(isX_0 + itX_1) \\
&= \mathsf{E} \exp \{ is(X_0 - \xi_1) + i(s+t)\xi_1 + it\zeta_1 \} \\
&= (1 - is/\beta)^{-\alpha\bar{\rho}} (1 - i(s+t)/\beta)^{-\alpha\rho} (1 - it/\beta)^{-\alpha\bar{\rho}} \qquad (5)
\end{aligned}
$$

and autocorrelation $\mathsf{Corr}\big(X_s, X_t\big) = \rho^{|s-t|}$. The process of passing from X_{t-1} to $\xi_t = X_{t-1} B_t$ is called *thinning*, so X_t is called the *thinned gamma process*. A similar construction is available for any ID marginal distribution.

2.3 Random Measure Gamma Process

Let $\mathcal{G}(dx\,dy)$ be a countably additive random measure that assigns independent random variables $\mathcal{G}(A_i) \sim \mathsf{Ga}(\alpha|A_i|, \beta)$ to disjoint Borel sets $A_i \in \mathcal{B}(\mathbb{R}^2)$ of finite area $|A_i|$ (this is possible by the Kolmogorov consistency conditions, and is illustrated in the Appendix) and, for $\lambda := -\log\rho$, consider the collection of sets:

$$G_t := \big\{(x, y) : \; x \in \mathbb{R}, \; 0 \le y < \lambda e^{-2\lambda|t-x|}\big\}$$

shown in Figure (1) whose intersections have area $|G_s \cap G_t| = e^{-\lambda|s-t|}$. For $t \in \mathcal{T} = \mathbb{R}$, set

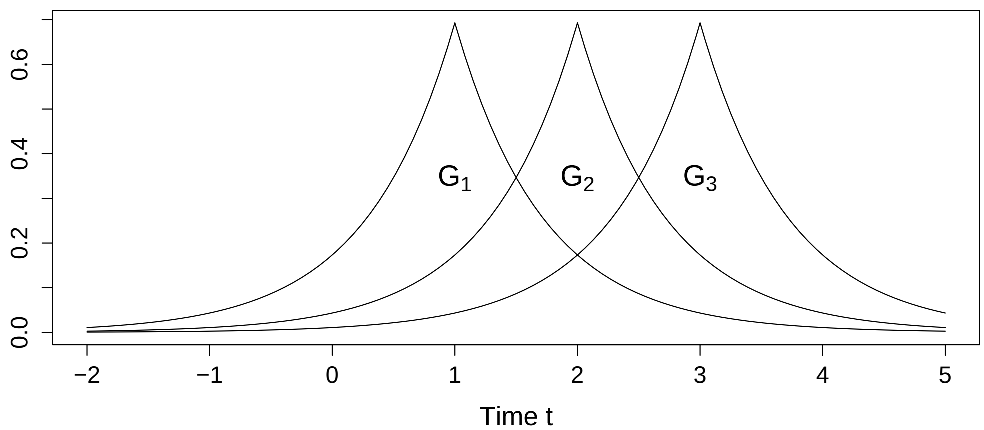

Figure 1: Random measure construction of process $X_t = \mathcal{G}(G_t)$

$$X_t := \mathcal{G}(G_t). \tag{6}$$

For any n times $t_1 < t_2 < \cdots < t_n$ the sets $\{G_{t_i}\}$ partition $\mathbb{R}^2$ into $n(n+1)/2$ sets of finite area (and one with infinite area, $(\cup G_{t_i})^c$), so each X_{t_i} can be written as the sum of some subset of $n(n+1)/2$ independent Gamma random variables. In particular, any $n = 2$ variables X_s and X_t can be written as

$$X_s = \mathcal{G}(G_s \backslash G_t) + \mathcal{G}(G_s \cap G_t), \qquad X_t = \mathcal{G}(G_t \backslash G_s) + \mathcal{G}(G_s \cap G_t)$$

just as in the thinning approach of Section (2.2), so both 1-dimensional and 2-dimensional marginal distributions for the random measure process coincide with those for the thinning process. Again the joint chf is

$$\begin{aligned}\chi(s, t) &= \mathsf{E}\exp(isX_0 + itX_1) \\ &= (1 - is/\beta)^{-\alpha\bar\rho}(1 - i(s+t)/\beta)^{-\alpha\rho}(1 - it/\beta)^{-\alpha\bar\rho}\end{aligned} \tag{7}$$

and the autocorrelation is $\mathsf{Corr}\big(X_s, X_t\big) = \exp\big(-\lambda|s-t|\big)$ or, for integer times, $\rho^{|s-t|}$ for $\rho := \exp(-\lambda)$. The distribution for consecutive *triplets* differs from those of the Thinned Gamma Process, however, an illustration that the thinning process is Markov but the random measure is not. The Random Measure process does feature infinitely-divisible (ID) marginal distributions of all orders, while the thinned process does not.

2.4 The Markov change-point Gamma Process

Let $\{\zeta_n : \ n \in \mathbb{Z}\} \overset{\text{iid}}{\sim} \mathsf{Ga}(\alpha, \beta)$ be iid Gamma random variables and let N_t be a standard Poisson process indexed by $t \in \mathbb{R}$ (so $N_0 = 0$ and $(N_t - N_s) \sim \mathsf{Po}(t-s)$ for all $-\infty < s < t < \infty$, with independent increments), and set

$$X_t := \zeta_n, \qquad n = N_{\lambda t}.$$

Then each $X_t \sim \mathsf{Ga}(\alpha, \beta)$ and, for $s, t \in \mathbb{R}$, X_s and X_t are either identical (with probability $\rho^{|s-t|}$) or independent— reminiscent of a Metropolis MCMC chain. The chf is

$$\begin{aligned}\chi(s,t) &= \mathsf{E}\exp(isX_0 + itX_1)\\ &= \rho\big(1 - i(s+t)/\beta\big)^{-\alpha} + \bar{\rho}(1 - is/\beta)^{-\alpha}(1 - it/\beta)^{-\alpha}\end{aligned} \tag{8}$$

and once again the marginal distribution is $X_t \sim \mathsf{Ga}(\alpha, \beta)$ and the autocorrelation function is $\mathsf{Corr}\big(X_s, X_t\big) = \rho^{|s-t|}$.

2.5 The Squared O-U Gamma Diffusion

Let $\{Z_i\} \overset{\text{iid}}{\sim} \mathsf{OU}(\lambda/2, 1)$ be independent Ornstein-Uhlenbeck velocity processes, mean-zero Gaussian processes with covariance $\mathsf{Cov}\big(Z_i(s), Z_j(t)\big) = \exp\big(-\frac{\lambda}{2}|s-t|\big)\delta_{ij}$, and set

$$X_t := \frac{1}{2\beta}\sum_{i=1}^{n} Z_i(t)^2$$

for $n \in \mathbb{N}$ and $\beta \in \mathbb{R}_+$. Note $Z_i(t) \sim \mathsf{No}(0,1)$, so $\mathsf{E}Z_i(t)^2 = 1$ and $\mathsf{E}Z_i(t)^4 = 3$; it follows that $\mathsf{E}Z_i(s)^2 Z_i(t)^2 = 1 + 2\exp(-\lambda|s-t|)$. Then $X_t \sim \mathsf{Ga}(\alpha, \beta)$ for $\alpha = n/2$, with $\mathsf{E}X_s = \alpha/\beta$ and

$$\begin{aligned}\mathsf{E}X_s X_t &= \frac{1}{4\beta^2}\left\{n\big(1 + 2\exp(-\lambda|s-t|)\big) + n(n-1)\right\}\\ &= \frac{\alpha^2 + \alpha\exp\big(-\lambda|s-t|\big)}{\beta^2}\end{aligned}$$

so the autocovariance is $\mathsf{Cov}\big(X_s, X_t\big) = \frac{\alpha}{\beta^2}\, e^{-\lambda|s-t|}$ and the autocorrelation at integer times is $\mathsf{Corr}\big(X_s, X_t\big) = \rho^{|s-t|}$ for $\rho := \exp(-\lambda)$. The chf at consecutive integer times is

$$\begin{aligned}\chi(s,t) &= \mathsf{E}\exp(isX_0 + itX_1)\\ &= \big(1 - i(s+t)/\beta - st(1-\rho)/\beta^2\big)^{-\alpha},\end{aligned} \tag{9}$$

distinct from (4), (5)=(7), and (8), so this process is new. Itô's formula is used in (Wolpert, 2011) to show that X_t has stochastic differential equation (SDE) representation

$$X_t = X_0 - \int_0^t 2\lambda\big(X_s - \alpha/\beta\big)\,ds + 2\sqrt{\lambda/\beta}\int_0^t \sqrt{X_s}\,dW_s \tag{10}$$

and hence has generator $\mathfrak{A}\phi(x) = (\partial/\partial\epsilon)\mathsf{E}[\phi(X_{t+\epsilon}) \mid X_t = x]\big|_{\epsilon=0}$ given by

$$\mathfrak{A}\phi(x) = -2\lambda(x - \alpha/\beta)\phi'(x) + (2\lambda/\beta)x\phi''(x), \tag{11}$$

which we will use to distinguish this process from that of Section (2.6). While the construction above required half-integer values for α, (9) is positive-definite and the SDE (10) has a unique strong solution for all $\alpha > 0$, so a time-reversible stationary Markov diffusion process exists with this distribution. Cox *et al.* (1985, Eqn (17)), building on (Feller, 1951), found the transition kernel for this process:

$$f(y, t \mid x, s) = ce^{-u-v}\left(\frac{v}{u}\right)^{(\alpha-1)/2} I_{(\alpha-1)}\big(2\sqrt{uv}\big),$$

where

$$c := \beta/\big[1 - e^{-2\lambda|t-s|}\big], \qquad u := cye^{-2\lambda|t-s|}, \qquad v := cx.$$

where $I_q(x)$ is the modified Bessel function of the first kind of order q. With this one can construct likelihood functions and generate posterior samples, MLEs, *etc.* for the parameters of this process. Cox *et al.* show that the process X_t is strictly positive at all times if $\alpha \geq 1$, but occasionally reaches zero for $\alpha < 1$.

2.6 Continuously Thinned Gamma Process

Pick a large integer n and set $\epsilon := 1/n$, $q := \exp(-\lambda\epsilon)$, and $p := 1 - q = \lambda\epsilon + o(\epsilon)$. Draw $X_0 \sim \mathsf{Ga}(\alpha, \beta)$ and, for integers $i, j \in \mathbb{N}$, draw independently

$$\zeta_i \sim \mathsf{Ga}(\alpha p, \beta) \qquad b_j \sim \mathsf{Be}(\alpha p, \alpha q).$$

Set:

$$\begin{aligned}
X_0 &= X_0 \\
X_\epsilon &= X_0(1 - b_1) &&+ \zeta_1 \\
X_{2\epsilon} &= X_\epsilon\,(1 - b_2) &&&&+ \zeta_2 \\
&= X_0(1 - b_1)(1 - b_2) &&+ \zeta_1(1 - b_2) &&+ \zeta_2 \\
X_{3\epsilon} &= X_0(1 - b_1)(1 - b_2)(1 - b_3) &&+ \zeta_1(1 - b_2)(1 - b_3) &&+ \zeta_2(1 - b_3) + \zeta_3
\end{aligned}$$

and, in general,

$$X_{k\epsilon} = X_0 \prod_{j=1}^{k}(1 - b_j) + \sum_{i=1}^{k}\left\{\zeta_i \prod_{i<j\leq k}(1 - b_j)\right\}. \tag{12a}$$

In the limit as $n \to \infty$ and $k\epsilon \to t$ the products converge to the product integral of the beta process introduced by Hjort (1980, §3) (and described lucidly by Thibaux and Jordan, 2007, §2) and the sum to an ordinary gamma stochastic integral,

$$X_t = X_0 \prod_{s\in(0,t]} [1 - dB(s)] + \int_0^t \left\{ \prod_{s\in(r,t]} [1 - dB(s)] \right\} \zeta(dr), \tag{12b}$$

where $\zeta(dr) \sim \mathsf{Ga}\big(\alpha\lambda\, dr, \beta\big)$ is a Gamma random measure and $B(s) \sim \mathsf{BP}\big(\alpha, \lambda\, ds\big)$ is a Beta process, *i.e.*, an SII Lévy process with Lévy measure

$$\nu_B(du) = \lambda\alpha\, u^{-1}(1-u)^{\alpha-1}\, \mathbf{1}_{\{0<u<1\}}\, du$$

with constant "concentration function" $\alpha(s) \equiv \alpha$ and translation-invariant "base measure" $\lambda(ds) = \lambda ds$. The product integral can be written as a ratio

$$\prod_{s\in(r,t]} [1 - dB(s)] = \frac{1 - F(t)}{1 - F(r)}$$

where $F(t) = \prod_{s\in(t,\infty)} [1 - dB(s)]$ satisfies

$$\frac{dF(t)}{1 - F(t)} = dB(t), \qquad B_t = \int_{(0,t]} \frac{dF(s)}{1 - F(s)}. \tag{13}$$

2.6.1 Generator

The Gamma process of Eqn (12b) is stationary and Markov, with generator

$$\begin{aligned} \mathfrak{A}\phi(x) = &\int_0^\infty \big[\phi(x+u) - \phi(x)\big]\, \alpha\lambda u^{-1} e^{-\beta u}\, du \\ &+ \int_0^x \big[\phi(x-u) - \phi(x)\big]\, \alpha\lambda u^{-1}(1 - u/x)^{\alpha-1}\, du \end{aligned} \tag{14}$$

Because this differs from (11) (and in particular because it is a non-local operator, showing X_t has jumps), this process is new. Once again the one-dimensional marginal distributions are $X_t \sim \mathsf{Ga}(\alpha, \beta)$ and the autocorrelation is $\mathsf{Corr}(X_s, X_t) = \exp\big(-\lambda|s-t|\big)$.a

3 Discussion

We have now constructed six distinct processes that share the same univariate marginal distribution and autocorrelation function, but which all differ in their n-variate marginal distributions for $n \geq 3$. Some are Markov, some not; some are time-reversible, some not; some have ID marginal distributions of all orders, some don't. Similar methods can be used to construct $\mathsf{AR}(1)$-like processes with *any* ID marginal distribution, such as those listed on $p.$ 10; only in the two cases of Gaussian and Poisson do all these constructions coincide. Many of these are useful for modeling time-dependent phenomena whose dependence falls off over time, but for which traditional Gaussian methods are unsuitable because of heavy tails, or integer values, or positivity, or for other reasons. I know of very little work (yet!) exploring inference for processes like these; a beginning appears in (Wang, 2013, §3). I have written code in `R` to generate samples from each of these six processes; available on request.

Appendix

Proposition 1 (Walker (2000))**.** *The innovations ζ_t in Eqn (3) can be constructed successively as follows:*

$$\lambda_t \sim \mathsf{Ga}(\alpha, 1), \qquad N_t \mid \lambda_t \sim \mathsf{Po}\big(\tfrac{1-\rho}{\rho}\lambda_t\big), \qquad \zeta_t \mid N_t \sim \mathsf{Ga}\big(N_t, \tfrac{\beta}{\rho}\big).$$

Proof. For $\omega \in \mathbb{R}$,

$$\begin{aligned}
\mathsf{E}e^{i\zeta_t\omega} \, |N_t &= (1 - i\omega\rho/\beta)^{-N_t} \\
\mathsf{E}e^{i\zeta_t\omega} \mid \lambda_t &= \sum_{n=0}^{\infty}(1 - i\omega\rho/\beta)^{-n}\big(\tfrac{1-\rho}{\rho}\lambda_t\big)^n \exp\big(-\tfrac{1-\rho}{\rho}\lambda_t\big)/n! \\
&= \exp\Big(i\frac{(1-\rho)\omega}{\beta - i\rho\omega}\lambda_t\Big) \\
\mathsf{E}e^{i\zeta_t\omega} \quad &= \Big[1 - i\frac{(1-\rho)\omega}{\beta - i\rho\omega}\Big]^{-\alpha} = \Big[\frac{\beta - i\omega}{\beta - i\rho\omega}\Big]^{-\alpha}.
\end{aligned}$$

□

Poisson and Gamma SII Processes

The chf for a Poisson random variable $X \sim \mathsf{Po}(\nu)$ is

$$\chi_X(\theta) = \mathsf{E}[e^{i\theta X}] = \sum_{k=0}^{\infty} e^{i\theta k}\left\{\frac{\nu^k}{k!}e^{-\nu}\right\} = e^{(e^{i\theta}-1)\nu}$$

so for any $u \in \mathbb{R}$ the re-scaled random variable $Y := uX$ has chf

$$\begin{aligned}
\chi_{uX}(\theta) &= \mathsf{E}[e^{i\theta uX}] = \chi_X(u\theta) \\
&= e^{(e^{i\theta u}-1)\nu}
\end{aligned}$$

and a linear combination $Y := \sum u_j X_j$ of independent $X_j \sim \mathsf{Po}(\nu_j)$ has chf

$$\begin{aligned}
\chi_Y(\theta) &= \prod_j \left\{e^{(e^{i\theta u_j}-1)\nu_j}\right\} \\
&= \exp\left\{\sum_j (e^{i\theta u_j} - 1)\nu_j\right\} && \text{(15a)} \\
&= \exp\left\{\int_{\mathbb{R}} (e^{i\theta u} - 1)\nu(du)\right\} && \text{(15b)}
\end{aligned}$$

for the discrete measure

$$\nu(du) = \sum \nu_j \delta_{u_j}(du)$$

that assigns mass ν_j to each point u_j, provided the sum in (15a) converges. Of course the sum converges if it has only finitely-many terms, or even if there are infinitely-many with $\sum \nu_j < \infty$ (because $|e^{i\theta u} - 1| \le 2$), but that condition isn't actually necessary. Since also $|e^{i\theta u} - 1| \le |\theta u|$, the random variable Y will be well-defined and finite provided

$$\sum_j \big(1 \wedge |u_j|\big)\nu_j < \infty \tag{16a}$$

or, in integral form,

$$\int_{\mathbb{R}} \big(1 \wedge |u|\big)\nu(du) < \infty. \tag{16b}$$

A random variable with chf of form (15a) for a sequence satisfying (16a) is called a "compound Poisson" distribution; one with the more general chf of form (15b) for a measure satisfying (16b) is called Infinitely Divisible, or ID.

ID Distributions

For any σ-finite measure satisfying (16b) it's easy to make a stochastic process with stationary independent increments of the more general form of (15b), beginning a Poisson random measure $\mathcal{N}(du\,ds)$ on $\mathbb{R} \times \mathbb{R}_+$ with intensity measure $\mathsf{E}\mathcal{N}(du\,ds) = \nu(du)\,ds$:

$$X_t := \iint_{\mathbb{R}\times(0,t]} u\mathcal{N}(du\,ds). \tag{17}$$

This is a right-continuous independent-increment nondecreasing process that begins at $X_0 = 0$ and has jumps $\Delta_t = [X_t - X_{t-}] \equiv [X_t - \lim_{s\nearrow t} X_s]$ of magnitudes $\Delta_t \in E$ at rate $\nu(E)$ for any Borel $E \subset \mathbb{R}$. The Poisson process itself is the special case where $\nu(E) = \nu\mathbf{1}_{\{1\in E\}}$, with jumps of magnitude $\Delta_t = 1$ at rate $\nu \in \mathbb{R}_+$.

Khinchine and Lévy (1936) showed that a random variable Y has a chf of the form (15b) if and only[1] if, for every $n \in \mathbb{N}$, one can write $Y = \zeta_1 + \cdots \zeta_n$ as the sum of n iid random variables ζ_j. This property is called "Infinite Divisibility" (abbreviated ID), and the processes we have constructed whose increments have this property are called "SII" processes for their stationary independent increments. Examples of ID distributions (or SII processes) and their Lévy measures include:

Poisson	$\mathsf{Po}(\lambda)$	$\nu(du) = \lambda\delta_1(du)$
Negative Binomial	$\mathsf{NB}(\alpha, p)$	$\nu(du) = \sum_{k\in\mathbb{N}} \alpha\frac{q^k}{k}\delta_k(du), \qquad q := (1-p)$
Gamma	$\mathsf{Ga}(\alpha, \beta)$	$\nu(du) = \alpha u^{-1} e^{-\beta u}\mathbf{1}_{\{u>0\}}\,du$
α-Stable	$\mathsf{St}_0(\alpha, \beta, \gamma, \delta)$	$\nu(du) = \frac{\alpha\gamma}{\pi}\Gamma(\alpha)\sin\frac{\pi\alpha}{2}\|u\|^{-\alpha-1}(1+\beta\,\mathrm{sgn}\,u)\,du$
Symmetric α-Stable	$\mathsf{S}\alpha\mathsf{S}(\alpha, \gamma)$	$\nu(du) = \frac{\alpha\gamma}{\pi}\Gamma(\alpha)\sin\frac{\pi\alpha}{2}\|u\|^{-\alpha-1}\,du$
Cauchy	$\mathsf{Ca}(\delta, \gamma)$	$\nu(du) = \frac{\gamma}{\pi}\|u\|^{-2}\,du.$

[1] For nonnegative random variables this is true as stated, but a slightly more general form is necessary for real-valued ID random variables, with a condition on $\nu(du)$ somewhat weaker than (16b) (see (18c))—if you get interested, ask me about "compensation". This is needed for the $\mathsf{Ca}(\delta, \gamma)$ example and, for $\alpha \ge 1$, the $\mathsf{St}_0(\alpha, \beta, \gamma, \delta)$ and $\mathsf{S}\alpha\mathsf{S}(\alpha, \gamma)$ examples below.

The defining condition for a random variable Y to be "Infinitely Divisible" (ID) is that for each $n \in \mathbb{N}$ there must exist iid random variables $\{\zeta_j : 1 \le j \le n\}$ such that Y and $\sum_{j=1}^n \zeta_j$ have the same distribution. This is clearly equivalent to the condition that every power $\chi^\alpha(\omega)$ of the characteristic function $\chi(\theta) := \mathsf{E}\exp(i\theta Y)$ must *also* be a characteristic function (*i.e.*, must be positive-definite) for each inverse integer $\alpha = 1/n$, because we can just take $\{\zeta_j\}$ to be iid with chf $\chi^{1/n}(\omega)$ and verify that their sum has chf $\chi(\omega)$. Less obvious but also true is that Y is ID if and only if $\chi^\alpha(\omega)$ is positive-definite for *all* real $\alpha > 0$, and even less obvious is the Khinchine and Lévy theorem that χ must take the specific form

$$\chi(\theta) = \exp\left\{ i\theta\delta - \theta^2\sigma^2/2 + \int_{\mathbb{R}} \left[e^{i\theta u} - 1\right] \nu(du) \right\} \tag{18a}$$

for some $\delta \in \mathbb{R}$, $\sigma^2 \ge 0$, and Borel measure ν satisfying $\nu(\{0\}) = 0$ and (16b) or, a little more generally, the form

$$\chi(\theta) = \exp\left\{ i\theta\delta - \theta^2\sigma^2/2 + \int_{\mathbb{R}} \left[e^{i\theta u} - 1 - i\theta h(u)\right] \nu(du) \right\} \tag{18b}$$

for any bounded function h that satisfies $h(u) = u + O(u^2)$ near $u \approx 0$ (like $\arctan u$ or $u\mathbf{1}_{\{|u|<1\}}$ or $u/(1+u^2)$) and a Borel measure ν satisfying the weaker restriction

$$\int_{\mathbb{R}} \left(1 \wedge u^2\right)\nu(du) < \infty. \tag{18c}$$

Some properties of ID distributions beyond our scope, but covered in (Steutel and van Harn, 2004) (see also (Bose *et al.*, 2002)), include:

Theorem 1 (S&vH, Thm 2.13)**.** *Let $\chi(\theta)$ be the chf of an Infinitely Divisible distribution. Then $(\forall\theta \in \mathbb{R})\,\{\chi(\theta) \ne 0\}$. Also, if $\chi(\theta)$ is analytic in some open domain $\Omega \subset \mathbb{C}$, then $(\forall\theta \in \Omega)\,\{\chi(\theta) \ne 0\}$. Thus, ID chfs do not vanish on $\mathbb{R}$ or anywhere in $\mathbb{C}$ that $\chi(\theta)$ is analytic.*

Theorem 2 (S&vH, Thm 9.8)**.** *Let X be an infinitely divisible random variable that is not normal or degenerate. Then the two-sided tail of X satisfies*

$$\lim_{x\to\infty} \frac{-\log \mathsf{P}[|X| > x]}{x \log x} = c$$

for a number $0 \le c < \infty$ given by $c^{-1} = \max[\nu(\mathbb{R}_+), \nu(\mathbb{R}_-)]$. An ID random variable that is not degenerate has a normal distribution if and only if the same limit is $c = \infty$.

Theorem 3 (S&vH, Prop 2.3)**.** *No non-degenerate bounded random variable is ID*

This one's easy enough to prove. If $\|X\|_\infty = B < \infty$ and X has the same distribution as $\sum_{j=1}^n \zeta_j$ for iid $\{\zeta_j\}$ then $\|\zeta_j\|_\infty = B/n$ and $\sigma^2 := \mathsf{V}(X) = n\mathsf{V}(\zeta_j) \le n\mathsf{E}\zeta_j^2 \le B^2/n$, so $\sigma^2 = 0$ and X must be degenerate.

Gamma Variables & Processes

The Gamma distribution $X \sim \mathsf{Ga}(\alpha, \beta)$ with mean α/β and variance α/β^2 has chf

$$\begin{aligned}\chi_X(\theta) &= \mathsf{E}[e^{i\theta X}] \\ &= \int_0^\infty e^{i\theta x} \left\{ \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \right\} dx \\ &= \big(1 - i\theta/\beta\big)^{-\alpha} \\ &= \exp\left\{-\alpha \log(1 - i\theta/\beta)\right\} \\ &= \exp\left\{ - \int_0^\infty \big(e^{i\theta u} - 1\big) \alpha u^{-1} e^{-\beta u}\, du \right\},\end{aligned}$$

exactly of the form of (15b) with Lévy measure

$$\nu(du) = \alpha u^{-1} e^{-\beta u} \mathbf{1}_{\{u>0\}}\, du.$$

This measure, while infinite, does satisfy condition (16b):

$$\begin{aligned}\int_{\mathbb{R}} \big(1 \wedge |u|\big) \nu(du) &= \int_0^1 \big(|u|\big) \alpha u^{-1} e^{-\beta u}\, du + \int_1^\infty \big(1\big) \alpha u^{-1} e^{-\beta u}\, du \\ &\le \int_0^\infty \alpha e^{-\beta u}\, du = \alpha/\beta < \infty.\end{aligned}$$

Thus gamma-distributed random variables are ID and the SII gamma *process*

$$X_t = \iint_{\mathbb{R}\times(0,t]} u \mathcal{N}(du\, ds)$$

has infinitely-many non-negative "jumps" $\Delta_t = X_t - X_{t-}$ in any time interval $a < t \le b$. Their sum $\sum\{\Delta_t : \ a < t \le b\} = X_b - X_a$ is finite, however, with probability distribution

$$X_b - X_a \sim \mathsf{Ga}\big(\alpha(b-a),\ \beta\big).$$

The gamma *random measure* $\mathcal{G}(dx\, dy) \sim \mathsf{Ga}(\alpha\, dx\, dy,\ \beta)$ of Section (2.3) has a similar construction,

$$\mathcal{G}(A) = \iiint_{\mathbb{R}\times A} u \mathcal{N}(du\, dx\, dy)$$

as the sum of the heights u_j of a Poisson cloud of points (u_j, x_j, y_j) for which $(x_j, y_j) \in A$. Wolpert and Ickstadt (1998) show how to simulate such random measures very efficiently, drawing the jumps $\{u_j\}$ in monotone decreasing order.

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