Some Notes on Gamma Processes

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February 10, 2017

1 Gamma Distribution

An exponential random variable $X \sim \mathsf{Ex}(\beta)$ with rate β has mean $\mu = 1/\beta$, complimentary CDF

$$\bar{F}(t) = \mathsf{P}[X > t] = e^{-\beta t}, \qquad t \geq 0,$$

and hence density function $f(t) = \lambda \exp(-\lambda t)\mathbf{1}_{\{t>0\}}$ and characteristic function (chf)

$$\xi_1(\omega) = \mathsf{E}[e^{i\omega X}] = (1 - i\omega/\beta)^{-1}.$$

It follows that the sum $Y = \sum_{j \leq \alpha} X_j$ of $\alpha \in \mathbb{N}$ iid random variables $X_j \sim \mathsf{Ex}(\beta)$ has chf

$$\xi_\alpha(\omega) = \mathsf{E}[e^{i\omega Y}] = (1 - i\omega/\beta)^{-\alpha} \tag{1}$$

the αth power of $\xi_1(\omega)$. A bit of algebra and induction show that the distribution has pdf

$$\begin{aligned} f(x \mid \alpha, \beta) &= \frac{\beta^\alpha}{(\alpha-1)!}\, x^{\alpha-1} e^{-\beta x} \mathbf{1}_{\{x>0\}} \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)}\, x^{\alpha-1} e^{-\beta x} \mathbf{1}_{\{x>0\}}. \end{aligned} \tag{2}$$

It turns out that $f(x \mid \alpha, \beta)$ of (2) is a valid pdf with corresponding chf $\xi_\alpha(\omega)$ of (1) for *all* $\alpha > 0$, not only integers. The distribution is called the "gamma distribution with shape α and rate β," denoted by $\mathsf{Ga}(\alpha, \beta)$. Other parametrizations are possible, and appear at times in the literature— the most common alternatives are "shape, scale," the pair α, s for scale $s = 1/\beta$, and "mean, variance", the pair $\mu = \alpha/\beta = \alpha s$ and $\sigma^2 = \alpha/\beta^2 = \alpha s^2$. The choice (α, β) is most convenient for Bayesian modeling and inference.

For integral α, $\mathsf{Ga}(\alpha, \beta)$ is also called the "Erlang distribution," and can be interpreted as the length of time until the αth event occurs for a Poisson process with rate β. The special case $\alpha = 1$ is just the exponential distribution $\mathsf{Ga}(1, \beta) = \mathsf{Ex}(\beta)$. For half-integers $\alpha = \nu/2$ with $\beta = 1/2$, $\mathsf{Ga}(\nu/2, 1/2) = \xi^2_\nu$ is the "chi squared distribution with ν degrees of freedom."

1.1 Properties of Gamma Distributions

Let $\{X_j \sim \mathsf{Ga}(\alpha_j, \beta)\}$ be independent for $1 \le j \le n$, all with the same rate parameter $\beta > 0$ but perhaps with different shape parameters $\alpha_j > 0$. Set $X_+ := \sum_j X_j$, their sum, and set $Y_j := X_j/X_+$ for $1 \le j \le n$. Then $X_+ \sim \mathsf{Ga}(\alpha_+, \beta)$ also has a gamma distribution with shape $\alpha_+ = \sum_j \alpha_j$ the sum of the shapes, and the same rate parameter β, while the vector $Y = (Y_1, \cdots, Y_n)$ has the n-variate Dirichlet distribution $Y \sim \mathsf{Di}(\alpha)$ with parameter vector $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n_+$, independent of X_+:

$$X_+ := \sum_j X_j \sim \mathsf{Ga}(\alpha_+, \beta) \qquad \perp\!\!\!\perp \qquad Y := X/X_+ \sim \mathsf{Di}(\alpha). \tag{3}$$

In particular, each $Y_j = X_j/X_+ \sim \mathsf{Be}(\alpha_j, \alpha_+ - \alpha_j)$ has a marginal beta distribution. For $n = 2$, the result can be written

$$X \sim \mathsf{Ga}(\alpha, 1) \perp\!\!\!\perp Y \sim \mathsf{Ga}(\beta, 1) \Leftrightarrow (X + Y) \sim \mathsf{Ga}(\alpha + \beta, 1) \perp\!\!\!\perp \frac{X}{X + Y} \sim \mathsf{Be}(\alpha, \beta) \tag{4}$$

Using this in reverse one can construct independent $X := ZB \sim \mathsf{Ga}(\alpha, 1)$ and $Y := Z(1 - B) \sim \mathsf{Ga}(\beta, 1)$ beginning with independent $Z \sim \mathsf{Ga}(\alpha + \beta, 1)$ and $B \sim \mathsf{Be}(\alpha, \beta)$.

The sum $Y = \sum_{j \le n} X_j$ of iid $X_j \sim \mathsf{Ga}(\alpha, \beta)$ has exactly the $\mathsf{Ga}(n\alpha, \beta)$ distribution and, by the Central Limit Theorem, approximately the $\mathsf{No}(n\alpha/\beta, n\alpha/\beta^2)$ distribution if n is large. It follows that

$$Y \sim \mathsf{Ga}(\alpha, \beta) \Rightarrow \frac{\beta Y - \alpha}{\sqrt{\alpha}} \approx \mathsf{No}(0, 1), \qquad \text{for large } \alpha.$$

2 The SII Gamma Process

In this section we construct and study a stochastic process ζ_t indexed by $t \in \mathbb{R}$ with the property that its increments $[\zeta(t_i) - \zeta(t_{i-1})]$ are independent with gamma $\mathsf{Ga}(\alpha(t_i - t_{i-1}), \beta)$ distributions. Because these distributions depend only on the length of the interval, and not its location, such process are said to have "stationary independent increments," or to be "SII processes" for short.

2.1 First Construction of Gamma Process

First let's construct $\zeta(t)$ for dyadic rational $t \in [0, 1]$. Set $\zeta_0 = 0$ and draw $\zeta_1 \sim \mathsf{Ga}(\alpha, \beta)$. For integers $0 \le j < \infty$ and $0 \le i < 2^j$, set $n := 2^j + i$ and draw independent random variables

$$B_n \sim \mathsf{Be}\big(2^{-j}, 2^{-j}\big), \qquad n = 2^j + i, \quad 0 \le i < 2^j.$$

For odd i, define $\zeta(t)$ for dyadic rational $t = i/2^j$ by

$$\zeta\Big(\frac{i}{2^j}\Big) = (1 - B_n)\,\zeta\Big(\frac{i-1}{2^j}\Big) + (B_n)\,\zeta\Big(\frac{i+1}{2^j}\Big)$$

(note that for odd i, $\zeta(t)$ for $t = (i \pm 1)/2$ will have been constructed at stage $(j-1)$ or earlier). By induction, and (4) with $X = \zeta\left(\frac{i}{2^j}\right) - \zeta\left(\frac{i-1}{2^j}\right)$ and $Y = \zeta\left(\frac{i+1}{2^j}\right) - \zeta\left(\frac{i}{2^j}\right)$, all increments of $\zeta(t)$ are independent and gamma-distributed with rate β and shape α times the length of the time intervals.

By construction the function $t \rightsquigarrow \zeta(t)$ is non-decreasing on the dyadic rationals $\mathbb{Q}$, so we may define

$$\zeta(t) = \inf\{\zeta(q) : \ q \in \mathbb{Q}, \ q \geq t\}$$

to complete the definition of $\zeta(t)$ for all real $0 \leq t \leq 1$. Repeating this construction independently for each interval $[n, n+1]$ for integers $n \in \mathbb{Z}$ and patching the processes together results in a construction of $\zeta(t)$ with the desired properties for all $t \in \mathbb{R}$.

From this construction it is clear that the paths of $\zeta(t)$ are *right* continuous, but it isn't obvious whether or not they are in fact continuous. The similar construction of Brownian motion leads to a path-continuous process; the similar construction of a Poisson process does not. The second construction presented in Section (2.2) shows that almost-surely the paths of $\zeta(t)$ are discontinuous (on the left) at every point t.

2.2 Poisson Construction of Gamma Process

In this section we construct the SII gamma process as a stochastic integral of a Poisson random measure. First, a few preliminaries about such random measures.

2.2.1 Poisson Random Measures

For any σ-finite Borel measure $\nu(dx)$ on a complete separable metric space $\mathcal{X}$, there exists a Poisson random measure $\mathcal{N} \sim \mathsf{Po}\big(\nu(dx)\big)$ that assigns independent Poisson random variables $\mathcal{N}(A_j) \sim \mathsf{Po}(\lambda_j)$ to disjoint Borel sets $A_j \subset \mathcal{X}$ of finite measure $\lambda_j = \nu(A_j) < \infty$. For any simple function

$$\phi(x) = \sum_j a_j \mathbf{1}_{A_j}(x)$$

in L_1 one can define the stochastic integral

$$\mathcal{N}[\phi] = \int_{\mathcal{X}} \phi(x) \mathcal{N}(dx) := \sum_j a_j \mathcal{N}(A_j),$$

a random variable with characteristic function, mean, and variance

$$\begin{aligned}\mathsf{E}\exp\big(i\omega\mathcal{N}[\phi]\big) &= \prod_j \exp\left\{(e^{i\omega a_j}-1)\nu(A_j)\right\} \\ &= \exp\Big\{\int_{\mathcal{X}}(e^{i\omega\phi(x)}-1)\nu(dx)\Big\} \qquad (5a)\end{aligned}$$

$$\mathsf{E}\mathcal{N}[\phi] = \int_{\mathcal{X}}\phi(x)\nu(dx) \qquad (5b)$$

$$\mathsf{V}\mathcal{N}[\phi] = \int_{\mathcal{X}}\phi^2(x)\nu(dx) \qquad (5c)$$

The continuous linear mapping $\phi \mapsto \mathcal{N}[\phi]$ can be extended from the simple functions to all of $L_1\big(\mathcal{X},\nu(dx)\big)$ by (5b). In fact the mapping can be extended further than this, to the Musielak-Orlicz (M-O) space (Musielak, 1983),

$$\mathcal{M} := \Big\{\phi : \int \big(1\wedge|\phi(x)|\big)\,\nu(dx) < \infty\Big\}, \qquad (6)$$

still with chf given by (5a) since $|e^{i\omega\phi(x)}-1| \le (|\omega|\vee 2)(1\wedge|\phi(x)|)$ for all $x\in\mathcal{X}$ and $\omega\in\mathbb{R}$.

2.2.2 Constructing the SII gamma process

Denote a σ-finite measure on the on the complete separable metric space $\mathcal{X} = \mathbb{R}_+\times\mathbb{R}$ by

$$\nu(du\,ds) = \alpha u^{-1}e^{-\beta u}\,du\,ds \qquad (7)$$

In this context, $\nu(du\,ds)$ is called the "Lévy measure" for the process $\zeta(t)$ defined below. Let $\mathcal{N}(du\,ds) \sim \mathsf{Po}\big(\nu(du\,ds)\big)$ be a Poisson random measure on $\mathcal{X}$ with mean ν. For $t\in\mathbb{R}$, set

$$\phi_t(u,s) = \begin{cases} u\mathbf{1}_{\{(0,t]\}}(s) & t>0 \\ u\mathbf{1}_{\{(t,0]\}}(s) & t\le 0\end{cases}$$

and define a random process by

$$\zeta(t) = \mathcal{N}[\phi_t] = \iint_{\mathbb{R}_+\times(0,t]} u\,\mathcal{N}(du\,ds) \qquad (8)$$

for $t>0$, with a similar expression for $t\le 0$. By (5a) the chf for an increment $[\zeta(t)-\zeta(s)]$ for $-\infty < s < t < \infty$ is

$$\begin{aligned}\mathsf{E}\exp\Big(i\omega\big(\mathcal{N}[\phi_t-\phi_s]\big)\Big) &= \exp\Big\{\int_{\mathbb{R}_+\times\mathbb{R}}(e^{i\omega u\mathbf{1}_{\{(s,t]\}}(z)}-1)\alpha u^{-1}e^{-\beta u}du\,dz\Big\} \\ &= \exp\Big\{\alpha(t-s)\int_{\mathbb{R}}(e^{i\omega u}-1)u^{-1}e^{-\beta u}du\Big\} \\ &= \big(1-i\omega/\beta\big)^{-\alpha(t-s)},\end{aligned}$$

so increments $[\zeta_t - \zeta_s]$ have $\mathsf{Ga}\big(\alpha(t-s), \beta\big)$ distributions. Increments over disjoint intervals are independent because $\mathcal{N}(A_j)$ are independent for sets $A_j \subset \mathbb{R}_+ \times (t_{j-1}, t_j]$ for any increasing sequence t_j.

From (8) we get a very clear picture of the sample paths for the process $\zeta(t)$. If we enumerate all the (random) mass points (u_i, s_i) of $\mathcal{N}(du\,ds)$, then $\zeta(t)$ increases by an amount u_i as t increases past each s_i. The $\{s_i\}$ are dense in $\mathbb{R}$, so $\zeta(t)$ is never constant and in fact has infinitely-many jump increases in every nonempty time interval $(s, t]$, but the increases are summable in any bounded time interval. The number of jumps of magnitude exceeding any threshold $\epsilon > 0$ in a time interval $(s, t]$ has distribution

$$\begin{aligned} J &:= \mathcal{N}\big((\epsilon, \infty) \times (s, t]\big) \\ &\sim \mathsf{Po}\big((t-s)\alpha \mathrm{E}_1(\beta\epsilon)\big), \end{aligned}$$

where $\mathrm{E}_1(z)$ is Gauss' exponential integral function (Abramowitz and Stegun, 1964, §5.1)

$$\mathrm{E}_1(z) = \int_z^\infty u^{-1} e^{-u}\, du, \qquad z > 0.$$

This is finite for $\epsilon > 0$, so only finitely-many jumps will exceed any $\epsilon > 0$ in a finite time interval. The probability that the maximum jump in that interval $U_{[1]}$ will be smaller than some number $u > 0$ is identical to the probability that $\mathcal{N}(du\,ds)$ puts zero points in the set $(u, \infty) \times (s, t]$, an easy Poisson calculation:

$$\mathsf{P}[U_{[1]} \le u] = \mathsf{P}\Big(\mathcal{N}\big((u, \infty) \times (s, t]\big) = 0\Big) = \exp\big(-(t-s)\alpha \mathrm{E}_1(\beta u)\big),$$

from which the pdf for $U_{[1]}$ is easily available. A similar approach will give the joint pdf for the largest k jumps $U_{[1]}, \cdots, U_{[k]}$ for any $k \in \mathbb{N}$.

2.3 Stochastic Integrals

One can construct stochastic integrals of L_1 simple functions $\psi(t) = \sum a_j \mathbf{1}_{(t_{j-1}, t_j]}(t)$ on $\mathbb{R}$ with respect to the SII gamma process, just as we did for Poisson random measures earlier:

$$\zeta[\psi] = \int \psi(t) \zeta(dt) = \sum_j a_j [\zeta(t_j) - \zeta(t_{j-1})]$$

then extend by continuity to L_1 or beyond that to the M-O space

$$\begin{aligned} \mathcal{M} &:= \Big\{\psi : \int_{\mathbb{R}} \big(1 \wedge |u\psi(s)|\big)\, \nu(du\,ds) < \infty\Big\} \\ &= \Big\{\psi : \int_{\mathbb{R}} \big(1 - e^{-1/|\psi(s)|}\big)\, \psi(s)\, ds + \int_{\mathbb{R}} \mathrm{E}_1\big(1/|\psi(s)|\big)\, ds < \infty\Big\}. \end{aligned}$$

Any $\psi \in L_1(ds)$ is also in $\mathcal{M}$, but so is (for example) $\psi(s) = |s|^{-\gamma}$ for any $\gamma > 1$, although such a function is not integrable in any neighborhood of zero and so is not in $L_1 \subsetneq \mathcal{M}$.

From this stochastic integral, or from the Poisson representation (8), we can construct a gamma random measure $\zeta(dt)$ assigning to disjoint Borel sets $B_j \subset \mathbb{R}$ independent gamma-distributed random variables $\zeta(B_j) \sim \mathsf{Ga}\big(\alpha|B_j|, \beta\big)$ with shape proportional to the Lebesgue measure $|B_j|$ of B_j:

$$\begin{aligned}\zeta(B) = \zeta[\mathbf{1}_B] \quad &= \int_{\mathbb{R}} \mathbf{1}_B(t)\,\zeta(dt) \\ = \mathcal{N}[u\mathbf{1}_B] &= \int_{\mathbb{R}_+\times\mathbb{R}} u\mathbf{1}_B(s)\,\mathcal{N}(du\,ds).\end{aligned}$$

2.4 SII Gamma Random Fields

The construction of a gamma random measure in Section (2.3) can be done in $\mathbb{R}^d$ just as easily as $\mathbb{R}^1$, or on manifolds or graphs or other objects. For $\mathbb{R}^d$, for example, begin with a Poisson random measure $\mathcal{N}(du\,dx) \sim \mathsf{Po}\big(\nu(du\,dx)\big)$ on $\mathbb{R}_+ \times \mathbb{R}^d$ with mean

$$\nu(du\,dx) = \alpha u^{-1} e^{-\beta u}\,du\,dx,$$

where now $x \in \mathbb{R}^d$ is a vector, and set

$$\zeta(B) = \mathcal{N}[u\mathbf{1}_B] = \int_{\mathbb{R}_+\times\mathbb{R}^d} u\mathbf{1}_B(s)\,\mathcal{N}(du\,dx)$$

to again have a random measure, now on $\mathbb{R}^d$, assigning to disjoint Borel sets B_j independent gamma-distributed random variables $\zeta(B_j) \sim \mathsf{Ga}\big(\alpha|B_j|, \beta\big)$. Using such a random measure one can build LARK semiparametric regression models (Wolpert et al., 2011; Wolpert and Ickstadt, 1998a) of the form

$$f(x) = \int k(x,y)\zeta(dy)$$

for kernel functions $k(x,y)$. Gamma random measures on a graphical representation of a road network were by Best et al. (2000) to model the uncertain amount of combustion biproducts generated by automobile, coach, and lorry traffic in a Bayesian epidemiological analysis of the effects of road pollution on respiratory disease rates in a midlands English town.

3 Approximations & Implementation

The mathematical construction of (8) can't be implemented precisely, because $\zeta(t)$ is represented as the sum of infinitely-many non-zero terms. Here we present two efficient approaches to approximating $\zeta(t)$ with finite sums

3.1 Truncation

One approach is to select a small threshold $\epsilon > 0$ and include in (8) only those finitely-many mass points (u, s) with $u \geq \epsilon$:

$$\zeta^\epsilon(t) = \mathcal{N}^\epsilon[\phi_t] = \iint_{(\epsilon,\infty)\times(0,t]} u\,\mathcal{N}(du\,ds) \tag{9}$$

where

$$\mathcal{N}^\epsilon(du\,ds) \sim \mathsf{Po}\big(\alpha u^{-1} e^{-\beta u} \mathbf{1}_{\{u>\epsilon\}}\,du\,ds\big).$$

The number $J_\epsilon = \mathcal{N}\big((\epsilon,\infty)\times(0,t]\big)$ of terms included is Poisson distributed with mean

$$\mathsf{E}J_\epsilon = t\alpha \mathrm{E}_1(\beta\epsilon) \leq t\alpha \log(1 + 1/\beta\epsilon). \tag{10}$$

This leads to truncation approximations for the SII gamma process, stochastic integral, and random measure:

$$\zeta^\epsilon(t) = \int_{(\epsilon,\infty)\times(0,t]} u\,\mathcal{N}(du\,ds) \quad \zeta^\epsilon[\psi] = \int_{(\epsilon,\infty)\times\mathbb{R}} u\psi(s)\,\mathcal{N}(du\,ds) \quad \zeta^\epsilon(B) = \int_{(\epsilon,\infty)\times B} u\,\mathcal{N}(du\,ds). \tag{11}$$

3.1.1 Truncation Error

The truncation error $\Delta_\epsilon(t) = [\zeta(t) - \zeta^\epsilon(t)]$ also has a stochastic integral representation,

$$\Delta_\epsilon(t) = \iint_{(0,\epsilon]\times(0,t]} u\,\mathcal{N}(du\,ds),$$

nonnegative with mean and variance

$$\mathsf{E}\Delta_\epsilon(t) = \iint_{(0,\epsilon]\times(0,t]} u\,\nu(du\,ds) = \mu_\epsilon t, \qquad \mu_\epsilon := (\alpha/\beta)[1 - e^{-\beta\epsilon}] \tag{12a}$$

$$\mathsf{V}\Delta_\epsilon(t) = \iint_{(0,\epsilon]\times(0,t]} u^2\,\nu(du\,ds) = \sigma_\epsilon^2 t, \qquad \sigma_\epsilon^2 := (\alpha/\beta^2)[1 - (1+\beta\epsilon)e^{-\beta\epsilon}], \tag{12b}$$

each of which converges to zero at rate $\asymp |\epsilon|$ as $\epsilon \to 0$. These give the pointwise bounds

$$\mathsf{E}|\Delta_\epsilon(t)|^2 \leq (t^2\mu_\epsilon^2 + t\sigma_\epsilon^2) \quad \leq \epsilon^2\alpha t(1+\alpha t)$$
$$\mathsf{P}\big[|\Delta_\epsilon(t)| > c\big] \leq (t^2\mu_\epsilon^2 + t\sigma_\epsilon^2)/c^2 \leq \epsilon^2\alpha t(1+\alpha t)/c^2$$

for each t and $c > 0$, but we can do something better than this by using a little martingale theory. The SII process $M_t := \Delta_\epsilon(t) - t\mu_\epsilon$ is an L_2 martingale with previsible quadratic variation (Protter, 1990, §III.5)

$$\langle M\rangle_t = t\sigma_\epsilon^2,$$

so for each $c, T > 0$ Doob's martingale maximal inequality (Doob, 1990, §VII.3) gives

$$\mathsf{P}\Big[\sup_{0\le t\le T} |\Delta_\epsilon(t) - t\mu_\epsilon| > c\Big] \le T\sigma_\epsilon^2/c^2 \le T\alpha\epsilon^2/c^2. \tag{13}$$

Together, Equations (10) and (12) or (13) can guide the selection of ϵ to attain a balance between a uniform stochastic bound on the truncation error $\Delta_\epsilon(t)$ and a stochastic bound on the computational expense J_ϵ.

3.2 Inverse Lévy Measure Algorithm

Wolpert and Ickstadt (1998a,b) introduced an efficient method of generating mass points for processes like these in monotone decreasing order, making it possible to choose the threshold dynamically or to ensure a non-stochastic bound for J.

View the number of J_u of jumps of magnitude exceeding $u > 0$ on a bounded interval $[0, T]$, for an SII Gamma process ζ_t, as a random function of a *decreasing* u. This process has independent Poisson-distributed increments, with (by (10)) mean $\mathsf{E}[J_u - J_v] = t\alpha[\mathrm{E}_1(\beta u) - \mathrm{E}_1(\beta v)]$. By a monotone decreasing time change $s \mapsto u(s) := {\mathrm{E}_1}^{-1}(s/\alpha T)/\beta$, the inverse function to the Lévy measure $\nu\big((u,\infty)\big) \mapsto s$, we can construct a standard unit-rate Poisson process

$$N_s = J_{u(s)} \sim \mathsf{Po}(s)$$

indexed by $s \ge 0$. If we denote by τ_n the event times of that process, the nth-largest jump of the gamma process is $u(\tau_n)$, or

$$u_n = {\mathrm{E}_1}^{-1}(\tau_n/\alpha T)/\beta, \qquad n \in \mathbb{N}. \tag{14a}$$

The jump *times* s_n are independent of these, distributed

$$s_n \overset{\text{iid}}{\sim} \mathsf{Un}(0, T). \tag{14b}$$

This leads to the ILM approximations for the SII gamma process, stochastic integral, and random measure of

$$\zeta_n(t) = \sum_{i=1}^n u_j \mathbf{1}_{\{s_j \le t\}} \qquad \zeta_n[\psi] = \sum_{i=1}^n u_j \psi(s_j) \qquad \zeta_n(B) = \sum_{i=1}^n u_j \mathbf{1}_B(s_j). \tag{15}$$

4 Extension: Gamma-Stable

The Lévy measure $\nu(du\,ds)$ of (7) is the special case $\gamma = 0$ of a larger family of measures,

$$\nu(du\,ds) = \alpha u^{\gamma-1} e^{-\beta u}\,du\,ds$$

which includes both the gamma (with $\gamma = 0$) and the skewed α-Stable (with $\beta = 0$), as well as something new (when both $\beta > 0$ and $\gamma > 0$). Some of its properties are explored in class notes (Wolpert, 2012) from a Duke stochastic processes course.

5 Stationary Gamma Processes

In some applications one wants a process X_t with $\mathsf{Ga}(\alpha, \beta)$ marginal distribution at each time t, with $(X_t - X_s)$ rather small for small $|t-s|$ and $X_s,\ X_t$ nearly independent for large $|t-s|$. The SII process of Section (2) doesn't serve that need. There exists essentially just one stationary Gaussian process X_t with autocorrelation $\mathsf{Cov}(X_s, X_t) = e^{-\lambda|t-s|}$, the Ornstein-Uhlenbeck Velocity process. There exists essentially just one stationary Poisson process X_t with autocorrelation $\mathsf{Cov}(X_s, X_t) = e^{-\lambda|t-s|}$, as well. However, there exist many different stationary processes with marginal $X_t \sim \mathsf{Ga}(\alpha, \beta)$ distribution and autocorrelation $\mathsf{Cov}(X_s, X_t) = e^{-\lambda|t-s|}$. Six of them are constructed and compared in class notes (Wolpert, 2014) from another Duke stochastic processes course. Code in `R` to generate all of them is available from the author.

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