STA732
Statistical Inference

Lecture 19: Least Favorable Distributions in Testing

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https://www2.stat.duke.edu/courses/Spring22/sta732.01/

Recap from Lecture 18

- UMP may fail to exist in two-sided testing $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$
- Method of Undetermined Multipliers
- UMP in two-sided testing $H_0 : \theta \leq \theta_0$ or $\theta \geq \theta_2$ vs $H_1 : \theta_1 < \theta < \theta_2$

So far, we have discussed out UMP in 1-param testing (simple vs simple, one-sided, two-sided) for simple distributions

Goal of Lecture 19

1. Least favorable distributions
2. Applications
    - Testing in the presence of nuisance parameters
    - Nonparametric testing in quality checking

Chap. 3.8 of Lehmann and Romano

We have seen the method of undetermined multipliers as a way to deal with composite hypotheses. Least favorable distributions is another method to reduce composite nulls to a simple one.

Least favorable distributions

Recall: the generic strategy to find UMP

What was the strategy that worked in one sided testing?

1. **Reduce the composite alternative to a simple alternative**: If H_1 is composite, fix $\theta_1 \in \Omega_1$ and test the null hypothesis against the simple alternative $\theta = \theta_1$. (Hope that doesn't depend on θ_1!)
2. **Collapse the composite null to a simple null**: If H_0 is composite, collapse the null hypothesis to a simple one
    - by reasoning that it only depends on a few points in Ω_0
    - or by averaging over the null space Ω_0 (this lecture)
3. **Apply Neyman-Pearson Lemma in simple vs simple**: if the resulting test does not depend on θ_1, then it will be UMP for H_0 vs H_1

Why are we talking about least favorable distributions?

We talked about least favorable prior in minimax estimation

Since we only care about the worst-case risk, each Bayes estimator with some prior gives a lower bound for the worst-case risk. The least favorable prior gives the tightest lower bound. In case of no gap, it provides a way to find the minimax estimator.

In testing with composite nulls

$$H_0 : X \sim f_\theta, \theta \in \Omega_0$$

We only case about the worst power function on the null

$$\mathbb{E}_\theta \phi(x) \leq \alpha, \forall \theta \in \Omega_0$$

Let's put a prior on Ω_0 and see!

The composite nulls vs simple alternative testing

Consider

$$
\begin{aligned}
H_0 :& X \sim f_\theta, \theta \in \Omega_0 \\
H_1 :& X \sim g,
\end{aligned}
$$

where g is known.

Impose a prior Λ on Ω_0 and consider the new hypothesis

$$H_\Lambda : X \sim h_\Lambda(x) := \int_{\Omega_0} f_\theta(x) d\Lambda(\theta).$$

Let's test simple vs simple H_Λ vs H_1

How to pick prior? How does the simpler testing help to find UMP test?

Least favorable distributions

Let β_Λ be the power of the most powerful (MP) level-α test ϕ_Λ (i.e. LRT) for testing H_Λ vs H_1.

Def. Least favorable distributions

Λ is a least favorable distribution if $\beta_\Lambda \leq \beta_{\Lambda'}$ for any prior Λ'

Intuitively,

The least favorable prior puts weights on the most difficult (in the sense of getting high power) parameters in Ω_0.

Example

Let $X_1, \ldots, X_n$ be i.i.d. $\mathcal{N}(\theta, 1)$. We consider testing $H_0 : \theta \leq 0$ against $H_1 : \theta > 0$. Find a UMP test.
Show that the least favorable distribution puts the prior mass on 0.

Use least favorable prior to find UMP in composite nulls

Thm. 3.8.1 in Lehmann and Romano

Suppose ϕ_Λ is a MP level-α test for testing H_Λ against $H_1 : g$. If ϕ_Λ is level-α for the original hypothesis H_0 (i.e., $\mathbb{E}_{\theta_0}\phi_\Lambda(x) \leq \alpha, \forall\theta_0 \in \Omega_0$), then

1. The test ϕ_Λ is MP for $H_0 : \theta \in \Omega_0$ vs. H_1.
2. The distribution Λ is least favorable.

proof: just go through the assumptions

Applications

1. Testing in the presence of nuisance parameters

Let $X_1, \dots, X_n$ be i.i.d. $\mathcal{N}(\theta, \sigma^2)$, where both θ, σ^2 are unknown. We consider testing $H_0 : \sigma \leq \sigma_0$ against $H_1 : \sigma > \sigma_0$. Find a UMP test.

θ is the nuissance parameter.

Strategy outline

1. Fix a simple alterative (θ_1, σ_1)
2. Choose a prior Λ to collapse our null hypothesis over
   - It is clear that the prior on σ should be dirac on σ_0
   - How about prior on θ?

     The least favorable prior should make the alternative hypothesis hard to distinguish
3. Check the condition for Thm. 3.8.1, conclude it is UMP for composite null vs simple alternative
4. Check the test does not depend on the choice of (θ_1, σ_1)

proof:

2. Nonparametric testing in quality checking

Identical light bulbs have lifetime $X_1, \dots, X_n$ with an arbitrary distribution P over $\mathbb{R}$. Let u be a fixed threshold for a satisfactory lifetime and $\mathbb{P}(X \leq u)$ be the probability of a given light bulb being unsatisfactory. Given the data of sample lifetimes we may be interested in testing whether the probability of having an unsatisfactory light bulb is too large:

$$H_0 : \mathbb{P}(X \leq u) \geq p_0 \text{ vs. } H_1 : \mathbb{P}(X \leq u) < p_0$$

Here p_0 is a fixed quality parameter.

Strategy outline

0. Reparameterize the distribution P with P^- and P^+ being the conditional distributions of $X \mid X \leq u$ and $X \mid X > u$ respectively, and $p = \mathbb{P}(X \leq u)$
1. Fix a simple alternative (P^-, P^+, p_1) with $p_1 < p_0$
2. Choose a prior Λ
3. Check the conditions of Thm. 3.8.1, which is checking ϕ_Λ is level α for the composite null
4. Check that ϕ_Λ does not depend on the choice of alternative hypothesis.

proof:

Summary

- Introduced least favorable distributions as a technique to deal with composite nulls

What is next?

UMP with additional restrictions

- Method of Undetermined Multipliers applied to testing $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$ with power derivative constraint (Keener 12.6)
- UMPU: uniformly most powerful unbiased test

Thank you for attending
See you on Wednesday in Old Chem 025