STA732
Statistical Inference

Lecture 08: Equivariant estimation

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Recap from Lecture 07

1. Formulate equivariant estimation under location models
    - Three parts: family, loss, estimator
2. Formulate general equivariant estimation

Goal of Lecture 08

1. Maximal invariant statistic
2. Construct minimum risk equivariant (MRE) estimator
3. Pitman estimator of location
4. Properties of the MRE

Chap. 10.2 in Keener or Chap. 3 in Lehmann and Casella

Maximal invariant statistic

Q: what will be the form of MRE in equivariant location estimation?

We are back to the location estimation problem.

The general case is skipped

Location invariant function

A function h on $\mathbb{R}^n$ is called location invariant if $h(x + a) = h(x)$ for all $x \in \mathbb{R}^n, a \in \mathbb{R}$.

Example: A location invariant statistic

$$Y(X) = \begin{pmatrix} X_1 - X_n \\ \vdots \\ X_{n-1} - X_n \end{pmatrix}$$

is location invariant

In fact, any location invariant statistic is a function of Y

Suppose $h(X)$ is an arbitrary location invariant function, take $a = -X_n$, we have

$$h(X) = h(X - X_n\mathbf{1}) = h(Y_1, \dots, Y_{n-1}, 0).$$

For the above reason, Y is called maximal invariant

Y carries at least as much information about X an any other invariant statistics $h(X)$!

Every location equivariant estimator has a decomposition

Lem. Lehmann-Casella 3.1.6

If δ_0 is a location equivariant estimator, then any other estimator is location equivariant if and only if it can be written as

$$\delta(X) = \delta_0(X) - U(X)$$

where the statistic U is location invariant.

Remark: combined with the observation in the previous slide, the decomposition is $\delta(X) = \delta_0(X) - v(Y)$, where Y is maximal invariant.

proof: simply go through the definitions

Construct minimum risk equivariant (MRE) estimator

MRE estimators are constructed by conditioning on Y

Thm. 10.4 in Keener and 3.1.10 in Lehmann Casella

Consider equivariant estimation of a location parameter with an invariant loss ρ. Suppose δ_0 is an equivariant estimator with finite risk. Suppose for a.e. $y \in \mathbb{R}^{n-1}$, there is a value $v^* = v^*(y)$ that minimizes

$$\mathbb{E}_0 \left[\rho(\delta_0(X) - v) \mid Y = y \right]$$

over $v \in \mathbb{R}$. Then an MRE estimator is given by

$$\delta_0(X) - v^*(Y).$$

proof idea: 1. Use the decomposition of an equivariant estimator; 2. make appear the conditonal on Y and use tower property

Strategy to compute MRE

According to Thm 10.4, the strategy to compute MRE is

1. Find an equivariant estimator δ_0:
   $\delta_0(X) = X_n$ works, just need to check it has finite risk
2. Need the conditional distribution of X_n given Y
3. Minimize the expectation $\mathbb{E}_0 \left[\rho(\delta_0(X) - v) \mid Y = y\right]$

Compute MRE (1)

1. Take $\delta_0 = X_n$
2. The joint density of Y and X_n under P_0 is

$$f(y_1 + x_n, \dots, y_{n-1} + x_n, x_n)$$

   The marginal density of Y is

$$\int f(y_1 + t, \dots, y_{n-1} + t, t) dt$$

3. The expectation $\mathbb{E}_0 \left[\rho(\delta_0(X) - v) \mid Y = y \right]$ is

$$\frac{\int \rho(t - v) f(y_1 + t, \dots, y_{n-1} + t, t) dt}{\int f(y_1 + t, \dots, y_{n-1} + t, t) dt}$$

Compute MRE (2)

$$\min_{v \in \mathbb{R}} \frac{\int \rho(t-v) f(y_1+t, \dots, y_{n-1}+t, t) dt}{\int f(y_1+t, \dots, y_{n-1}+t, t) dt}$$

Since $y_i = x_i - x_n$, applying change of variables $t \leftarrow x_n - u$, we obtain the equivalent expression

$$\min_{v \in \mathbb{R}} \frac{\int \rho(x_n - v - u) f(x_1 - u, \dots, x_{n-1} - u, x_n - u) du}{\int f(x_1 - u, \dots, x_{n-1} - u, x_n - u) du}$$

Because our final MRE is $\delta^*(x) = x_n - v^*(y)$, δ^* is

$$\arg\min_{d} \frac{\int \rho(d - u) f(x_1 - u, \dots, x_{n-1} - u, x_n - u) du}{\int f(x_1 - u, \dots, x_{n-1} - u, x_n - u) du}$$

Pitman estimator of location

Pitman estimator of location under squared error loss

Under squared error loss $\rho(d-u) = (d-u)^2$, the MRE is unique and can be found explicitly

$$\delta^*(X) = \frac{\int u f(X_1 - u, \ldots, X_n - u) du}{\int f(X_1 - u, \ldots, X_n - u) du}$$

called Pitman estimator

Example: i.i.d. uniform with unknown mean

$X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} U(\theta - \frac{b}{2}, \theta + \frac{b}{2})$, b known.
Find MRE under squared error loss. Is it unbiased?

Example: i.i.d. exponential variables

Suppose $\epsilon_1, \dots, \epsilon_n$ are i.i.d. standard exponential variables. $X_i = \theta + \epsilon_i$. Determine the MRE of θ under squared error loss.

Properties of MRE

MRE under squared error loss for location family is unbiased

Lem. Lehmann-Casella 3.1.23

Under squared error loss and location family,

- If $\delta(X)$ is equivariant with constant bias $b \neq 0$, then $\delta(X) - b$ is unbiased and equivariant with smaller risk than $\delta(X)$
- MRE is unique and unbiased
- If UMVU exists and is location equivariant, then it is MRE

proof:

A comparison of MRE and uniform minimum risk unbiased (UMRU)

1. **Loss:**
   - MRE requires the loss to be invariant, and it is loss-dependent
   - For any convex loss, if complete sufficient statistic T exists, the UMVU and UMRU are constructed in the same way (Rao-Blackwellization)
2. **Admissibility:**
   - UMVU are not always admissible (often inadmissible in problems that lack symmetry or are in high dimension)
   - Pitman estimators are usually admissible (Stein, 1959)

Summary

- MRE can be constructed by conditioning on the maximal invariant statistic
- MRE has explicit form under squared error loss and location family: Pitman's estimator
- MRE vs. UMVU

What is next?

- Bayesian estimation

Thank you