STA732
Statistical Inference

Lecture 18+19: UMP in two-sided testing? + Least Favorable Distributions

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Recap from Lecture 17

- Introduced the concept of UMP tests
- LRT is UMP for one-sided testings in 1-param family with monotone likelihood ratios (which includes exponential family with natural parameter or strictly increasing $\eta(\theta)$)
- Reviewed p-values and the duality between testing and interval estimation

Goal of Lecture 18 + 19

1. A generic strategy for finding UMP for composite vs composite
2. UMP may fail to exist in two-sided testing $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$
3. Method of Undetermined Multipliers
4. UMP in two-sided testing $H_0 : \theta \leq \theta_0$ or $\theta \geq \theta_2$ vs $H_1 : \theta_1 < \theta < \theta_2$
5. Least favorable distributions

Chap. 12.5-12.7 of Keener or Chap. 3.6, 3.7, 3.8 of Lehmann and Romano

Where are we?

Types of optimality:

Point estimation	Hypothesis testing
Uniform (in general does not exist)	UMP
Restrict: UMVU, MRE	?
Global: Bayes, Minimax	?
Asymptotics	?

UMP has several cases to be resolved

- Simple vs. simple (Neyman-Pearson Lemma)
- One-sided (MLR OK)
- Two-sided (depends)
- General (may not exists)
- Two strategies to find UMP
  - Method of undetermined multipliers
  - Least favorable distributions

A generic strategy to find UMP

Recall: uniformly most powerful tests (UMP)

Def. Uniformly most powerful tests

A test ϕ^* with level α is called uniformly most powerful (UMP) if

$$\mathbb{E}_\theta \phi^* \geq \mathbb{E}_\theta \phi, \quad \forall \theta \in \Omega_1,$$

for all ϕ with level at most α.

It is equivalent to the following formulation

$$\begin{aligned} \max_{\phi} \quad & \mathbb{E}_{\theta_1} \phi \\ \text{s.t.} \quad & \mathbb{E}_{\theta_0} \phi \leq \alpha \end{aligned}$$

for every pair of $\theta_0 \in \Omega_0$ and $\theta_1 \in \Omega_1$

One generic strategy to find UMP

What was the strategy that worked in one sided testing?

1. **Reduce the composite alternative to a simple alternative**: If H_1 is composite, fix $\theta_1 \in \Omega_1$ and test the null hypothesis against the simple alternative $\theta = \theta_1$. (Hope that doesn't depend on θ_1!)
2. **Collapse the composite null to a simple null**: If H_0 is composite, collapse the null hypothesis to a simple one
    - by reasoning that it only depends on a few points in Ω_0
    - or by averaging over the null space Ω_0 (next half lecture)
3. **Apply Neyman-Pearson Lemma in simple vs simple**: if the resulting test does not depend on θ_1, then it will be UMP for H_0 vs H_1

Two-sided hypothesis testing

Two sided hypotheses can take two forms

- The case H_1 is two-sided (usual form)
  $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$
  or $H_0 : \theta_1 \leq \theta \leq \theta_2$ vs $H_1 : \theta > \theta_2$ or $\theta < \theta_1$
- The case H_0 is two-sided
  $H_0 : \theta \neq \theta_0$ vs $H_1 : \theta = \theta_0$
  or $H_0 : \theta \geq \theta_2$ or $\theta \leq \theta_1$ vs $H_1 : \theta_1 < \theta < \theta_2$

Application cases of two-sided hypotheses

- A new drug may be declared equivalent to the current standard drug if the difference in therapeutic effect is small, meaning θ is a small interval about 0.
- Determine whether a measuring instrument, for example a scale, is properly balanced.

UMP fails to exist in two-sided testing $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$

Suppose $X_1, \dots, X_n \sim \mathcal{N}(\theta, 1)$. Given the hypotheses $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$. Is there a UMP test ϕ^* of level-α?

Proof idea:

- Construct a UMP ϕ_1 for $H_0 : \theta = \theta_0$ vs $H_1' : \theta > \theta_0$, and another UMP ϕ_2 for $H_0 : \theta = \theta_0$ vs $H_1'' : \theta < \theta_0$.
- Show that ϕ^* cannot coincide with both, there is issue at θ_0

the LRT is not UMP; however it is a UMP unbiased (UMPU) test (to be discussed in a few lectures)

Method of Undetermined Multipliers

Motivation

In this proof of simple vs simple Neyman Pearson Lemma, the following proposition says that it suffices to maximize the unconstrained optimization problem with Langrange multiplier (k) in order to solve the constrained problem:

Prop 12.1 in Keener

Suppose $k \geq 0$, ϕ^* maximizes

$$\mathbb{E}_{\theta_1}\phi - k\mathbb{E}_{\theta_0}\phi$$

among all test functions, and $\mathbb{E}_{\theta_0}\phi^* = \alpha$. Then ϕ^* maximizes $\mathbb{E}_{\theta_1}\phi$ over all ϕ with level at most α

Now that the null hypothesis is divided into two parts, we need multiple multipliers!

Lem 3.6.1 in Lehmann and Romano

Let $F_1, \ldots, F_{m+1}$ be real-valued functions defined over a space U, and consider the problem of

$$\begin{aligned} \max \quad & F_{m+1}(u) \\ \text{s.t.} \quad & F_i(u) = c_i, \quad \forall i = 1, \ldots, m \end{aligned}$$

It is sufficient to find u^0 that satisfies the constraints and maximizes

$$F_{m+1}(u) - \sum_{i=1}^{m} k_i F_i(u)$$

for some undetermined multipliers $k_1, \ldots, k_m$.

Proof of Lem 3.6.1:

Methods of Undetermined Multipliers applied to testing (1)

We plan to apply the Methods of Undetermined Multipliers to the case U is the space of test functions ϕ:

$$F_i(\phi) = \int \phi(x) f_i(x) d\mu(x).$$

We want to

$$\begin{aligned} \max \quad & \int \phi(x) f_{m+1}(x) d\mu(x) \\ \text{s.t.} \quad & \int \phi(x) f_i(x) d\mu(x) = c_i, \quad \forall i = 1, \dots, m \end{aligned}$$

Methods of Undetermined Multipliers applied to testing (2)

According to Lem 3.6.1, we consider to maximize

$$F_{m+1}(\phi) - \sum_i k_i F_i(\phi) = \int \phi(x) \left(f_{m+1}(x) - \sum_{i=1}^{m} k_i f_i(x) \right) d\mu(x)$$

It is not hard to show (ignoring all regularity assumptions), the optimal solution should have the form

$$\phi(x) = \begin{cases} 1 & \text{if } f_{m+1}(x) > \sum_{i=1}^{m} k_i f_i(x) \\ 0 & \text{if } f_{m+1}(x) < \sum_{i=1}^{m} k_i f_i(x) \end{cases}$$

Finally, we choose k_i so that the constraints are all satisfied

Existence of ϕ^* in general space (convex and closed) requires some technical details, see Chapter 12.5 Keener

UMP in two-sided testing: H_0 two sided

In this section, we deal with two-sided testing $H_0 : \theta \leq \theta_1$ or $\theta \geq \theta_2$ vs $H_1 : \theta_1 < \theta < \theta_2$

Two-sided testing in the 1-param exponential family

Thm 3.7.1 in Lehmann and Romano

For testing the hypothesis $H_0 : \theta \leq \theta_1$ or $\theta \geq \theta_2 (\theta_1 < \theta_2)$ against the alternatives $H_1 : \theta_1 < \theta < \theta_2$ in the 1-param exponential family $h(x)e^{\eta(\theta)T(x) - B(\theta)}$ with η strictly increasing

1. There exists a UMP test given by

$$\phi(x) = \begin{cases} 1 & \text{when } C_1 < T(x) < C_2 (C_1 < C_2) \\ \gamma_i & \text{when } T(x) = C_i, i = 1, 2 \\ 0 & \text{when } T(x) < C_1 \text{ or } > C_2 \end{cases}$$

   with C and γ determined by $\mathbb{E}_{\theta_1} \phi(X) = \mathbb{E}_{\theta_2} \phi(X) = \alpha$.

2. This test minimizes $\mathbb{E}_\theta \phi(X)$ subject to $\mathbb{E}_{\theta_1} \phi(X) = \mathbb{E}_{\theta_2} \phi(X) = \alpha$ for all $\theta < \theta_1$ and $> \theta_2$.
3. For $0 < \alpha < 1$ the power function has a maximum at a point θ_0 between θ_1 and θ_2 and decreases strictly as θ tends away from θ_0, unless $\exists t_1, t_2$ s.t. $P_\theta \{T(X) = t_1\} + P_\theta \{T(X) = t_2\} = 1, \forall \theta$

Before we prove the theorem, we first show two useful ideas with sufficient statistics

- It suffices to restrict attention to tests based on sufficient statistics
- If the densities for X come from an exponential family, then the densities for sufficient statistics T will also be from an exponential family

It suffices to restrict attention to sufficient statistics

Thm. 12.17 in Keener

Suppose that T is sufficient for the model $\mathscr{P} = \{P_\theta : \theta \in \Omega\}$. Then for any test $\phi = \phi(X)$, the test

$$\psi = \psi(T) = \mathbb{E}_\theta[\phi(X) \mid T]$$

has the same power function as ϕ,

$$\mathbb{E}_\theta \psi(T) = \mathbb{E}_\theta \phi(X), \forall \theta \in \Omega$$

Proof:

$$\mathbb{E}_\theta \phi(X) = \mathbb{E}_\theta \mathbb{E}_\theta[\phi(X) \mid T] = \mathbb{E}_\theta \psi(T).$$

Density for sufficient statistics in exp family

Thm. 12.19 in Keener

If the distribution for X comes from an exponential family with densities

$$p_\theta(x) = h(x)e^{\eta(\theta)\cdot T(x) - B(\theta)}, \theta \in \Omega$$

then the induced distribution for $T = T(X)$ has density

$$q_\theta(t) = e^{\eta(\theta)\cdot t - B(\theta)}, \theta \in \Omega$$

with respect to some measure ν.

proof omitted

Proof of Thm 3.7.1 (1)

Proof outline

1.
   - Restrict attention to test based on sufficient statistics $T(X)$

   $$q_\theta(t) = e^{\eta(\theta)\cdot t - B(\theta)}$$

   - Apply Methods of Undetermined Multipliers to say the optimal test that maximizes $\mathbb{E}_{\theta'}\phi(X)$ subject to $\mathbb{E}_{\theta_1}\phi(X) = \mathbb{E}_{\theta_2}\phi(X) = \alpha$ ($\theta_1 < \theta' < \theta_2$) takes the form

   $$\phi(x) = \mathbf{1}_{\left\{k_1 e^{\eta(\theta_1)\cdot t - B(\theta_1)} + k_2 e^{\eta(\theta_2)\cdot t - B(\theta_2)} < e^{\eta(\theta')\cdot t - B(\theta')}\right\}}$$

   which has the form $a_1 e^{b_1 t} + a_2 e^{b_2 t} < 1$
   - Both a_1 and a_2 have to be positive, otherwise it won't be a two-sided test (then cannot satisfy the constraint)
   - Need to show it satisfies $\mathbb{E}_\theta \psi(T) \le \alpha$ for $\theta \le \theta_1$ and $\theta \ge \theta_2$, using Part 2.

Proof of Thm 3.7.1 (2)

Proof outline

2. • For $\theta' < \theta_1$. Apply Methods of Undetermined Multipliers to minimize $\mathbb{E}_{\theta'}\phi(X)$ subject to $\mathbb{E}_{\theta_1}\phi(X) = \mathbb{E}_{\theta_2}\phi(X) = \alpha$. show that the desired test has a rejection region of the form

$$a_1 e^{b_1 t} + a_2 e^{b_2 t} < 1,$$

which coincides with what is in Part 1. The optimal test is unique provied $P_\theta\{T = C_i\} = 0$

Proof of Thm 3.7.1 (3)

Proof outline

3. Intuitively we need a lemma similar to Cor 12.4 in Keener (says if a LRT has level-α, then its power is larger than α)
   - Suppose there exist three point $\theta' < \theta'' < \theta'''$ such that

$$\beta(\theta'') \leq \beta(\theta') = \beta(\theta''') = c$$

   - According to Part 1, the test maximizes $\mathbb{E}_{\theta''}\phi(X)$ subject to $\mathbb{E}_{\theta'}\phi(X) = \mathbb{E}_{\theta'''}\phi(X) = c$.
   - Use Corollary 3.6.1 to reach a contradiction

Cor 3.6.1 in Lehmann and Romano

Let $p_1, \ldots, p_m, p_{m+1}$ be probability densities, and let $0 < \alpha < 1$. Then $\exists$ a test ϕ s.t. $E_i\phi(X) = \alpha$ $(i = 1, \ldots, m)$ and $E_{m+1}\phi(X) > \alpha$, unless $p_{m+1} = \sum_{i=1}^{m} k_i p_i$, a.e.

proof omitted, by induction over m

Summary

Here is we can say about UMP (in 1-param testing) without additional constraints or restrictions

- **Simple vs simple**: LRT is UMP by Neyman-Pearson Lemma
- **One-sided**: with monotone Likelihood ratio, then LRT is UMP
- **Two-sided**
  - UMP may not exist in $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$
  - UMP exists for exp family in $H_0 : \theta \leq \theta_1$ or $\theta \geq \theta_2$ vs $H_1 : \theta_1 < \theta < \theta_2$
- **General**: UMP may not exist

Least favorable distributions

Recall: the generic strategy to find UMP

What was the strategy that worked in one sided testing?

1. **Reduce the composite alternative to a simple alternative**: If H_1 is composite, fix $\theta_1 \in \Omega_1$ and test the null hypothesis against the simple alternative $\theta = \theta_1$. (Hope that doesn't depend on θ_1!)
2. **Collapse the composite null to a simple null**: If H_0 is composite, collapse the null hypothesis to a simple one
   - by reasoning that it only depends on a few points in Ω_0
   - or by averaging over the null space Ω_0 (this half lecture)
3. **Apply Neyman-Pearson Lemma in simple vs simple**: if the resulting test does not depend on θ_1, then it will be UMP for H_0 vs H_1

Why are we talking about least favorable distributions?

We talked about least favorable prior in minimax estimation

Since we only care about the worst-case risk, each Bayes estimator with some prior gives a lower bound for the worst-case risk. The least favorable prior gives the tightest lower bound. In case of no gap, it provides a way to find the minimax estimator.

In testing with composite nulls

$$H_0 : X \sim f_\theta, \theta \in \Omega_0$$

We only case about the worst power function on the null

$$\mathbb{E}_\theta \phi(x) \leq \alpha, \forall \theta \in \Omega_0$$

Let's put a prior on Ω_0 and see!

The composite nulls vs simple alternative testing

Consider

$$
\begin{aligned}
H_0 \, :& X \sim f_\theta, \theta \in \Omega_0 \\
H_1 \, :& X \sim g,
\end{aligned}
$$

where g is known.

Impose a prior Λ on Ω_0 and consider the new hypothesis

$$
H_\Lambda : X \sim h_\Lambda(x) := \int_{\Omega_0} f_\theta(x) d\Lambda(\theta).
$$

Let's test simple vs simple H_Λ vs H_1

How to pick prior? How does the simpler testing help to find UMP test?

Least favorable distributions

Let β_Λ be the power of the most powerful (MP) level-α test ϕ_Λ (i.e. LRT) for testing H_Λ vs H_1.

Def. Least favorable distributions

Λ is a least favorable distribution if $\beta_\Lambda \leq \beta_{\Lambda'}$ for any prior Λ'

Intuitively,

The least favorable prior puts weights on the most difficult (in the sense of getting high power) parameters in Ω_0.

Example

Let $X_1, \ldots, X_n$ be i.i.d. $\mathcal{N}(\theta, 1)$. We consider testing $H_0 : \theta \leq 0$ against $H_1 : \theta > 0$. Find a UMP test.
Show that the least favorable distribution puts the prior mass on 0.

Use least favorable prior to find UMP in composite nulls

Thm. 3.8.1 in Lehmann and Romano

Suppose ϕ_Λ is a MP level-α test for testing H_Λ against $H_1 : g$. If ϕ_Λ is level-α for the original hypothesis H_0 (i.e., $\mathbb{E}_{\theta_0}\phi_\Lambda(x) \leq \alpha, \forall \theta_0 \in \Omega_0$), then

1. The test ϕ_Λ is MP for $H_0 : \theta \in \Omega_0$ vs. H_1.
2. The distribution Λ is least favorable.

proof: just go through the assumptions

Applications of least favorable distributions

Let $X_1, \dots, X_n$ be i.i.d. $\mathcal{N}\left(\theta, \sigma^2\right)$, where both θ, σ^2 are unknown. We consider testing $H_0 : \sigma \leq \sigma_0$ against $H_1 : \sigma > \sigma_0$. Find a UMP test.

θ is the nuissance parameter.

Strategy outline

1. Fix a simple alterative (θ_1, σ_1)
2. Choose a prior Λ to collapse our null hypothesis over
   - It is clear that the prior on σ should be dirac on σ_0
   - How about prior on θ?

     The least favorable prior should make the alternative hypothesis hard to distinguish
3. Check the condition for Thm. 3.8.1, conclude it is UMP for composite null vs simple alternative
4. Check the test does not depend on the choice of (θ_1, σ_1)

proof:

2. Nonparametric testing in quality checking

Identical light bulbs have lifetime $X_1, \dots, X_n$ with an arbitrary distribution P over $\mathbb{R}$. Let u be a fixed threshold for a satisfactory lifetime and $\mathbb{P}(X \leq u)$ be the probability of a given light bulb being unsatisfactory. Given the data of sample lifetimes we may be interested in testing whether the probability of having an unsatisfactory light bulb is too large:

$$H_0 : \mathbb{P}(X \leq u) \geq p_0 \text{ vs. } H_1 : \mathbb{P}(X \leq u) < p_0$$

Here p_0 is a fixed quality parameter.

Strategy outline

0. Reparameterize the distribution P with P^- and P^+ being the conditional distributions of $X \mid X \leq u$ and $X \mid X > u$ respectively, and $p = \mathbb{P}(X \leq u)$
1. Fix a simple alternative (P^-, P^+, p_1) with $p_1 < p_0$
2. Choose a prior Λ
3. Check the conditions of Thm. 3.8.1, which is checking ϕ_Λ is level α for the composite null
4. Check that ϕ_Λ does not depend on the choice of alternative hypothesis.

proof:

Summary

- Introduced least favorable distributions as a technique to deal with composite nulls

What is next?

UMP with additional restrictions

- Method of Undetermined Multipliers applied to testing $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$ with power derivative constraint (Keener 12.6)
- UMPU: uniformly most powerful unbiased test

Thank you