STA732
Statistical Inference

Lecture 20: UMP restricted to unbiased tests

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https://www2.stat.duke.edu/courses/Spring23/sta732.01/

Recap from Lecture 19

- Least favorable distributions as a way to reduce composite null to simple null

In general,

think Langrangian multipler for constrained optimization

think least favorable prior when dealing worst-case criteria!

Goal of Lecture 20

1. What to do when UMP does not exist
2. General strategies for uniformly most powerful unbiased (UMPU) tests
3. UMP with power derivative restriction

Chap. 12.5-7 of Keener or Chap. 4 of Lehmann and Romano

Beyond UMP

Types of optimality:

Point estimation	Hypothesis testing
Uniform (in general does not exist)	UMP
Restrict: UMVU, MRE	Chap 4-6 in Lehmann and Romano
Global: Bayes, Minimax	Chap 8
Asymptotics	Chap 11-13

Typicall approaches to go beyond UMP

Restrict to smaller class of test functions

- Unbiased test
- Invariance
- Monotonicity

Global measures

- Maximize the average power: put a prior on Ω_1
- Maximize worst case power: maximize the minimum power over Ω_1

Unbiased tests

Def. Unbiased test, 12.25 in Keener

Let $\alpha \in [0, 1]$. A test ϕ is unbiased level-α if

$$\beta_\phi(\theta) \leq \alpha, \forall \theta \in \Omega_0 \text{ and } \beta_\phi(\theta) \geq \alpha, \forall \theta \in \Omega_1$$

Remark

- Unbiasedness enforces the appealing property that the probability of rejection is greater under any alternative distribution than it is under any null distribution
- It is related to a special case of risk unbiasedness if we design the loss function L such that $L(\theta_0, \text{reject}) = 1 - \alpha$, $L(\theta_0, \text{accept}) = 0$, $L(\theta_1, \text{reject}) = 0$ and $L(\theta_1, \text{accept}) = \alpha$.

Restrict test with some invariance (not covered)

Let $X_1, \dots, X_n \overset{\text{iid}}{\sim} \mathcal{N}(\theta, \sigma^2)$ for σ, θ both unknown, and test $H_0 : \theta = 0$ versus $H_1 : \theta \neq 0$. For $i \in \{1, \dots, n\}$, let $X_i' = cX_i$ with $c > 0$. Then $\mathbb{E}(X_i') = \theta' = c\theta$. Since testing $\theta = 0$ is equivalent to testing $\theta' = 0$, it is natural to impose the invariance constraint

$$\forall c > 0 \quad \phi(X) = \phi(cX)$$

Such a test is unaffected by arbitrary rescaling of the data (which might occur when changing units from centimeters to meters).

Restrict to tests with monotonicity (not covered)

Let X, Y be independent, $X \sim \mathcal{N}(\theta_X, 1)$ and $Y \sim \mathcal{N}(\theta_Y, 1)$ for θ_X, θ_Y unknown, and test $H_0 : \theta_X \leq 0, \theta_Y \leq 0$.

A monotonicity restriction requires that if ϕ rejects upon observing (x, y), then it should also reject for (x', y') where $x' > x$ and $y' > y$.

General strategies for UMPU

Strategy outline

1. Prove that unbiasedness implies weaker constraints (α-similarity)
2. Fix an alternative hypothesis
3. Find a MP test ϕ under the weaker constraints (generalization of Neyman-Pearson lemma)
4. If ϕ does not depend on the alternative hypothesis, then it is UMP for the composite alternative under the weaker constraints
5. Show ϕ is UMP under the original constraint (unbiasedness).

Common boundary

Testing $H_0 : \theta \in \Omega_0$ vs $H_1 : \theta \in \Omega_1$. Ω_0 and Ω_1 are subsets of a Euclidean space. Let ω be the common boundary between Ω_0 and Ω_1:

$$w = \bar{\Omega}_0 \cap \bar{\Omega}_1$$

In words, ω is the intersection of the closures of Ω_0 and Ω_1

Examples of common boundary

Example 1

Testing $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$, then $\omega = \{\theta_0\}$

Example 2

Testing $H_0 : \theta_1 \leq \tilde{\theta}$ vs $H_1 : \theta_1 > \tilde{\theta}$ in the presence of nuisance parameters $(\theta_2, \ldots, \theta_{k+1})$, then

$$\omega = \left\{\theta \in \mathbb{R}^{k+1} : \theta_1 = \tilde{\theta}\right\}$$

α-similarity

Def. α-similarity, 4.1 in Lehmann and Romano

A test ϕ satisfying $\mathbb{E}_\theta \phi(X) = \alpha$ for all $\theta \in \omega$ is called α-similar on ω

Relation to unbiasedness

When $\beta_\phi(\theta)$ is continuous in θ, unbiasedness implies α-similarity on ω.

draw a picture

UMP among α-similar is sufficient for UMPU

Lem. 4.1.1 Lehmann and Romano

If $\theta \mapsto \beta_\phi(\theta)$ is continuous on Ω for all ϕ, and ϕ_0 is a UMP test among α-similar level-α tests, then ϕ_0 is also UMPU at level α

Proof: compare to the constant test

ϕ_0 is UMP among α-similar tests, it is at least as powerful as the constant test $\phi_\alpha(X) \equiv \alpha$.

UMPU in two-sided testing without nuisance params

Testing $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$. Suppose X is from a 1-param exp family

$$p_\theta(x) = h(x) \exp(\theta T(x) - A(\theta))$$

UMPU in two-sided testing without nuisance params

Testing $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$. Suppose X is from a 1-param exp family

$$p_\theta(x) = h(x) \exp(\theta T(x) - A(\theta))$$

- We know that no UMP test exist in the Gaussian case
- Assume ϕ is unbiased at level-α, then

$$\beta_\phi(\theta_0) = \mathbb{E}_{\theta_0} \phi(X) = \alpha$$

$$\beta_\phi(\theta_0) \leq \beta_\phi(\theta) \text{ for all } \theta \in \mathbb{R}$$

- If we further assume β_ϕ is differentiable, then the second point translate to

$$0 = \beta'_\phi(\theta_0) = \int \phi(x) \frac{d}{d\theta} p_{\theta_0}(x) d\mu(x)$$

To find UMPU in two-sided testing, we first find UMP with power derivative constraint

$$\max_{\phi} \beta_{\phi}(\theta') \quad \forall \theta' \in \Omega_1$$

$$\text{s.t. } \beta_{\phi}(\theta_0) = \alpha$$

$$\beta'_{\phi}(\theta_0) = 0$$

Method of undetermined multipliers allow us to deal with UMP problems with multiple constraints!

Proof for UMP with power derivative constraint in two-sided testing, 1-param exp family

- Fix a simple alternative $\theta' > \theta_0$
- Use method of undetermined multipliers to determine a rejection region for the simple vs simple testing
- Discuss the shape of the rejection region
- Find UMP test for $H_0 : \theta_0, H_1 : \theta' > \theta_0$
- Reverse the above argument to show the same test works for $H_0 : \theta_0, H_1 : \theta' < \theta_0$
- The test does not depend on the alternative, so UMP for the composite alternative

Recall: Methods of Undetermined Multipliers applied to testing (1)

We plan to apply the Methods of Undetermined Multipliers to the case U is the space of test functions ϕ:

$$F_i(\phi) = \int \phi(x) f_i(x) d\mu(x).$$

We want to

$$\begin{aligned} \max \quad & \int \phi(x) f_{m+1}(x) d\mu(x) \\ \text{s.t.} \quad & \int \phi(x) f_i(x) d\mu(x) = c_i, \quad \forall i = 1, \dots, m \end{aligned}$$

Recall: Methods of Undetermined Multipliers applied to testing (2)

According to Lem 3.6.1, we consider to maximize

$$F_{m+1}(\phi) - \sum_i k_i F_i(\phi) = \int \phi(x) \left(f_{m+1}(x) - \sum_{i=1}^{m} k_i f_i(x) \right) d\mu(x)$$

It is not hard to show (ignoring all regularity assumptions), the optimal solution should have the form

$$\phi(x) = \begin{cases} 1 & \text{if } f_{m+1}(x) > \sum_{i=1}^m k_i f_i(x) \\ 0 & \text{if } f_{m+1}(x) < \sum_{i=1}^m k_i f_i(x) \end{cases}$$

Finally, we choose k_i so that the constraints are all satisfied

Existence of ϕ^* in general space (convex and closed) requires some technical details, see Chapter 12.5 Keener

Conclude that the UMP test with with power derivative constraint is also UMPU

- First, $\phi_\alpha \equiv \alpha$ also satisfies the two constraints. Since ϕ is more powerful, then ϕ is unbiased.
- Second, all unbiased tests satisfy the two constraints. Conclude that ϕ is UMP among all level$-\alpha$ unbiased tests.

Example

Suppose $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma^2)$ with $H_0 : \sigma = \sigma_0$ vs $H_1 : \sigma \neq \sigma_0$

Summary

- UMPU exists in two sided testing without nuisance parameters
- UMPU via method of undetermined multipliers

What is next?

UMPU in multiparameter exp family

- Nuisance parameters
- the idea of conditioning

Thank you