STA732
Statistical Inference

Lecture 21: UMPU in multiparam exponential family

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Spring 2023

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https://www2.stat.duke.edu/courses/Spring23/sta732.01/

Recap from Lecture 20

- Unbiased tests
- UMPU for two-sided testing $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$

Recap on testing in one-param exponential family

The three cases of testing in one-param exponential family.

$$p_\theta(x) = h(x)\exp(\theta T(x) - A(\theta))$$

1. $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$
2. $H_0 : \theta \leq \theta_1$ or $\theta \geq \theta_2$ vs $H_1 : \theta_1 < \theta < \theta_2$
3. $H_0 : \theta_1 \leq \theta \leq \theta_2$ vs $H_1 : \theta < \theta_1$ or $\theta > \theta_2$
   similarly $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$

What about multi-param exponential family?

$$p_{\theta,\eta}(x) = h(x) \exp\left(\theta U(x) + \eta^\top T(x) - A(\theta, \eta)\right)$$

What can we say about the optimal tests?

1. $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$
2. $H_0 : \theta \leq \theta_1$ or $\theta \geq \theta_2$ vs $H_1 : \theta_1 < \theta < \theta_2$
3. $H_0 : \theta_1 \leq \theta \leq \theta_2$ vs $H_1 : \theta < \theta_1$ or $\theta > \theta_2$
   similarly $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$

Goal of Lecture 21

1. Conditional tests
2. UMPU for multi-param exponential family with nuisance param
3. Examples
    - Comparing two Poisson distributions
    - Testing Gaussian variance with unknown mean
    - Testing Gaussian mean with unknown variance

Chap. 13.1-3 of Keener

Conditional tests

Ways to deal with nuisance parameters

- Least favorable distributions
- Conditioning

Intuition about conditioning

$$p_{\theta,\eta}(x) = h(x) \exp\left(\theta U(x) + \eta^\top T(x) - A(\theta, \eta)\right)$$

Want to test $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$

- The boundary $\omega = \{(\theta, \eta) : \theta = \theta_0\}$
- On the boundary, θ is known, so $T(X)$ is sufficient for η
- The conditional $U(X) \mid T(X)$ has no η dependency on ω

Conditioning can eliminate the influence of nuisance parameters!

Strategy for conditional tests

1. Condition on $T(X) = t$
2. For each value of t, construct a "optimal conditional test" $\phi(u, t)$ which maximizes conditional power $\mathbb{E}_\theta[\phi(u, t) \mid T = t]$ $\forall \theta > \theta_0$
   and satisfies the conditional level α,
   $\mathbb{E}_\theta[\phi(u, t) \mid T = t] \leq \alpha, \forall \theta \leq \theta_0$
3. Check whether this test is UMPU at level α

Recall α-similarity

Def. α-similarity, 4.1 in Lehmann and Romano

A test ϕ satisfying $\mathbb{E}_\theta \phi(X) = \alpha$ for all $\theta \in \omega$ is called α-similar on ω

Try out the strategy for multi-param exp family?

- For each t, the one-param exp family of $U(X) \mid T(X) = t$ has MLR in $U(X)$, the "optimal conditional test" for fixed t should take the form

$$\phi(u,t) = \begin{cases} 1 & \text{if } u > c(t) \\ \gamma(t) & \text{if } u = c(t) \\ 0 & \text{otherwise} \end{cases}$$

with $c(t), \gamma(t)$ chosen to satisfy the conditional level constraint at the boundary θ_0 and $\theta \le \theta_0$

$$\mathbb{E}_{\theta_0}\left[\phi(U,T) \mid T = t\right] = \alpha \tag{1}$$

- Taking expectation on the power and level, deduce $\phi(U,T)$ is level α and UMP amongst level α tests satisfying (1). But (1) is more stringent than α-similarity!

UMPU for multi-param exponential family with nuisance param

We first need to inspect the relationship between (1) and α-similarity more carefully

Def. Neyman structure. 13.4 in Keener

Suppose T is sufficient for $\{P_\gamma : \gamma \in \omega\}$. A test ϕ has α-Neyman structure if ϕ satisfies

$$\mathbb{E}_\gamma \phi(X) \mid T(X) = \alpha, \text{ for a.e.} t, \forall \gamma \in \omega.$$

It is easy to show α-Neyman structure implies α-similarity:

Just take expectation, and by tower property.

α-Neyman structure under complete sufficient statistic

Thm. 13.5 in Keener

If T is complete and sufficient for $\{P_\gamma : \gamma \in \omega\}$, then everty α-similar test has α-Neyman structure with respect to T.

Neyman structure and similarity are equivalent whenever a complete sufficient statistic T exists!

Proof of Thm. 13.5:

Suppose α is α-similar

- Introduce $\Psi(T) = \mathbb{E}\left[\phi(X) - \alpha \mid T\right]$, which does not depend on γ because T is sufficient
- $\mathbb{E}_\gamma \Psi(T) = 0, \forall \gamma \in \omega$
- By completeness, $\Psi(T) = 0$! We obtain Neyman structure

Back to the multi-param exp family

$$p_{\theta,\eta}(x) = h(x) \exp\left(\theta U(x) + \eta^\top T(x) - A(\theta, \eta)\right)$$

If $T(X)$ is sufficient and complete on ω, we can

- Go from UMP among α-Neyman structure tests to UMP among α-similar tests
- Apply Lem. 4.1.1 Lehmann and Romano (or just compare to the constant test), UMP amont α-similar tests implies UMPU at level α.
- We can complete the proof for UMPU

Thm. UMPU in multi-param exp family

$$p_{\theta,\eta}(x) = h(x)\exp\left(\theta U(x) + \eta^\top T(x) - A(\theta,\eta)\right)$$

Thm 13.6 in Keener

Consider the above exponential family, if it is full rank and Ω is open, then ϕ_1 is UMPU test for $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$.

$$\phi_1(u,t) = \begin{cases} 1 & \text{if } u > c(t) \\ \gamma(t) & \text{if } u = c(t) \\ 0 & \text{otherwise} \end{cases}$$

Thm 13.6 also deals with UMPU for two-sided test (proof omitted)

Remarks about conditional tests

- **Proof summary:**
  - UMP among Neyman structure tests is easy to find if T is sufficient on the boundary, the optimal conditional test is reduced to one-param testing
  - If T is complete, then α-similar tests are equivalent to Neyman structure tests
- Conditional tests are easier to explicitly construct after observing the data

Examples

1. Compare two Poisson distributions

$X \sim$ Poisson(ν) and $Y \sim$ Poisson(μ) for X, Y independent. (May think X and Y as the number of successful recoveries from a disease under two different treatments)
Testing $H_0 : \mu \leq \nu$ vs $H_1 : \mu > \nu$

- it is equivalent to testing $H_0 : \log(\mu/\nu) \leq 0$ vs $H_1 : \log(\mu/\nu) > 0$
- so additional information in (μ, ν) is considered nuisance.

The joint density of (X, Y) is given by

$$\frac{1}{x!y!}\exp(-\mu - \nu)\exp(x\log(\nu) + y\log(\mu))$$
$$= \frac{1}{x!y!}\exp(-\mu - \nu)\exp\left(y\log(\mu/\nu) + (x + y)\log(\nu)\right)$$

Let $U = Y, T = X + Y$. And the natural parameters are $\theta = \log(\mu/\nu)$ and $\eta = \log(\nu)$, where η is nuisance param

form of the UMPU test?

2. Testing Gaussian variance with unknown mean

Let $X_1, \ldots, X_n$ be i.i.d. $\mathcal{N}(\mu, \sigma^2)$, where both μ, σ^2 are unknown. We consider testing $H_0 : \sigma \leq \sigma_0$ against $H_1 : \sigma > \sigma_0$. Find a UMPU test.

- We did find UMP in Lecture 19 with least favoroable distributions
- What does Basu's theorem say about $\bar{X}$ and $\sum_{i=1}^{n}(X_i - \bar{X})^2$?

3. Testing Gaussian mean with unknown variance

Let $X_1, \ldots, X_n$ be i.i.d. $\mathcal{N}(\mu, \sigma^2)$, where both μ, σ^2 are unknown. We consider testing $H_0 : \mu \leq \mu_0$ against $H_1 : \mu > \mu_0$. Find a UMPU test. Gives us t-test!

Summary

UMPU for multi-param exponential family with nuisance param, via conditional tests

- Construct conditional tests
- α-Neyman structure vs α-similarity

What is next?

- Testing in GLM

Thank you