STA113: Solutions for Homework 1

1. (a)

$$
\begin{aligned}
P(\text{at least 2 of 20 people have same b-day}) &= 1 - P(\text{none of 20 share same b-day}) \\
&= 1 - \frac{(365)(364)(363)\ldots(346)}{365^{20}} \\
&= 1 - \frac{1.037 \times 10^{51}}{365^{20}} \\
&= 1 - \frac{1.037 \times 10^{51}}{1.71 \times 10^{51}} \\
&= 1 - 0.5886 \\
&= 0.4114
\end{aligned}
$$

(b) Minitab exercise

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MTB > random 20 c1-c25;
SUBC> integers 1 365.
MTB > tally c1-c25.
```

C1	COUNT	C2	COUNT	C3	COUNT	C4	COUNT
13	1	4	1	18	1	26	1
53	1	42	1	33	1	41	1
55	1	77	1	38	1	54	1
71	1	98	1	55	1	73	1
126	1	104	1	97	2	110	1
147	1	135	1	129	1	121	1
153	1	149	1	135	1	156	1
162	1	156	1	161	1	172	1
179	1	162	1	167	1	193	1
194	1	177	1	173	1	198	1
205	1	178	1	183	1	234	1

225	1	186	1	185	1	266	1
227	1	204	1	249	1	268	1
232	1	207	1	251	1	273	1
265	1	240	1	260	1	293	1
274	1	243	1	277	1	300	1
292	1	248	1	320	1	305	1
311	1	269	1	334	1	317	1
327	1	290	1	350	1	324	1
355	1	361	1	N=	20	361	1
N=	20	N=	20			N=	20

C5	COUNT	C6	COUNT	C7	COUNT	C8	COUNT
1	1	5	1	23	1	2	1
74	1	19	1	74	1	16	1
102	1	84	1	90	1	33	1
122	1	126	1	100	1	41	1
143	1	136	1	108	1	69	1
154	1	156	1	115	1	108	1
157	1	171	1	187	1	117	1
166	1	172	1	252	1	180	1
209	1	180	1	258	1	195	2
252	1	189	1	264	1	214	1
253	1	193	1	270	1	225	1
257	1	195	1	271	1	281	1
260	1	215	1	279	1	293	1
274	1	221	1	295	1	309	1
277	1	245	1	309	1	324	1
286	1	256	1	320	1	335	1
287	1	346	2	330	1	336	1
324	1	360	1	354	1	347	1
326	1	363	1	359	1	362	1
330	1	N=	20	364	1	N=	20
N=	20			N=	20		

C9	COUNT	C10	COUNT	C11	COUNT	C12	COUNT
9	1	11	1	9	1	7	1
82	1	13	1	39	1	12	1
93	1	27	1	55	1	28	1

118	1	44	1	94	1	43	1
136	1	64	1	111	1	47	1
172	1	80	1	124	1	73	1
175	1	93	1	136	1	74	1
196	1	114	1	140	1	113	1
214	1	123	1	178	1	131	1
216	1	184	1	182	1	137	1
220	1	191	1	195	1	144	1
256	1	192	1	214	1	174	1
271	1	229	1	230	1	210	1
279	1	271	1	234	1	220	1
287	1	276	1	241	1	234	1
296	1	300	1	268	1	269	1
312	1	321	1	288	1	289	1
341	1	325	1	292	1	322	1
344	1	356	1	340	1	336	1
352	1	361	1	357	1	361	1
N=	20	N=	20	N=	20	N=	20

C13	COUNT	C14	COUNT	C15	COUNT	C16	COUNT
56	2	11	1	17	1	18	1
91	1	28	1	23	1	39	1
102	1	67	1	34	1	83	2
106	1	109	1	43	1	119	1
127	1	112	1	48	1	149	1
138	1	117	1	62	1	158	1
144	1	121	1	63	1	185	1
162	1	136	1	118	1	193	1
180	1	180	1	125	1	194	1
190	1	182	1	128	1	200	1
233	1	189	1	156	1	233	1
261	1	197	2	167	1	291	1
263	1	200	1	198	1	305	1
284	1	201	1	237	1	311	1
291	1	232	1	269	1	322	1
311	1	252	1	287	1	327	1
323	1	309	1	329	1	333	1

328	1	341	1	339	1	344	1
365	1	355	1	356	2	356	1
N=	20	N=	20	N=	20	N=	20

C17	COUNT	C18	COUNT	C19	COUNT	C20	COUNT
8	1	9	1	7	1	7	2
33	1	43	1	11	1	12	1
48	1	62	1	23	1	39	2
93	1	65	1	28	1	66	1
98	1	67	1	73	1	68	1
107	1	142	1	84	1	99	1
111	1	164	1	86	1	120	1
157	1	175	1	122	1	150	1
195	1	182	1	182	1	164	1
212	1	186	1	195	1	178	1
234	2	200	1	196	1	181	1
241	1	229	1	201	1	232	1
247	1	232	1	251	1	237	1
260	1	269	1	252	1	258	1
275	1	311	1	268	1	259	1
313	1	340	1	286	1	265	1
325	1	345	1	306	1	337	1
333	1	349	1	324	1	361	1
350	1	354	1	347	1	N=	20
N=	20	359	1	363	1		
		N=	20	N=	20		

C21	COUNT	C22	COUNT	C23	COUNT	C24	COUNT
25	1	2	1	6	1	2	1
48	1	45	1	11	1	34	1
71	1	65	1	29	1	43	1
104	1	87	1	41	1	51	1
115	1	146	1	58	1	111	1
135	1	155	1	69	1	156	1
142	1	169	1	111	1	166	1
168	1	170	1	171	1	167	1
187	1	188	1	184	1	173	1

202	1
205	1
238	1
254	1
264	1
265	1
267	1
292	1
323	2
353	1
N=	20

191	1
207	1
236	1
271	1
300	1
305	1
310	1
344	1
346	1
350	1
358	1
N=	20

188	1
205	1
208	1
222	1
226	1
264	1
283	1
311	1
319	1
339	1
349	1
N=	20

203	1
217	1
239	1
267	1
269	1
339	1
342	1
343	1
353	1
355	1
356	1
N=	20

C25	COUNT
14	1
20	1
22	1
32	1
37	1
45	1
108	2
112	1
124	2
149	1
154	1
158	1
207	1
220	1
239	1
271	1
303	1
305	1
N=	20

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MTB >
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$$\begin{aligned} x &= \#\text{ of groups (out of 25) of 20 people that had no matching b-days} \\ &= 14 \end{aligned}$$

probability estimate based on our estimation: p_e

$$
\begin{aligned}
p_e(\text{none of 20 share a b-day}) &= \frac{x}{25} \\
p_e(\text{at least 2 of 20 people share a b-day}) &= 1 - \frac{x}{25} \\
&= \frac{25 - x}{25} \\
&= \frac{11}{25} \\
&= 0.44
\end{aligned}
$$

2.

E = Engineering
P = Physics
F = Failure
W = Works
1 = 1st
2 = 2nd

example:

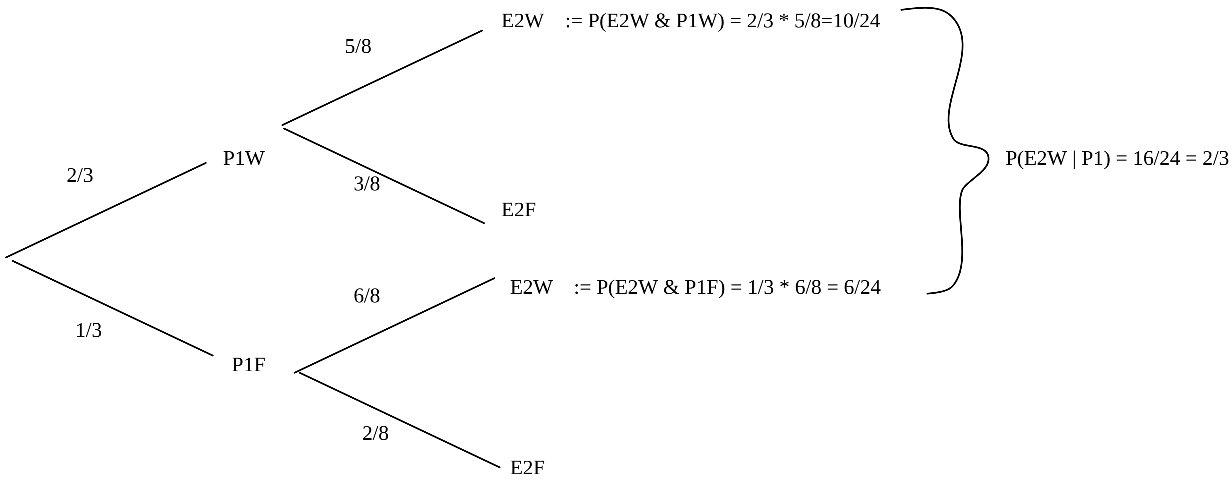
$$
\begin{aligned}
E1W &= \text{Engineering buys theirs first and it works} \\
P2 &= \text{Physics buys theirs second}
\end{aligned}
$$

9 routers, 3 are defective

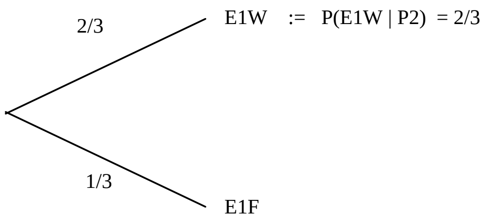
(a) $P(E2W|P1W) = \frac{5}{8}$

(b) $P(E2W|P1F) = \frac{6}{8}$

Physics buys theirs first:



Physics buys theirs second:



(c) Since $P(E2W|P1) = \frac{2}{3} = P(E1W|P2)$, it makes no difference if Physics goes first.

(d)

$$\begin{aligned} P(E1W|P2W) &= \frac{P(E1W \cap P2W)}{P(P2W)} \\ &= \frac{P(P2W|E1W)P(E1W)}{P(P2W)} \\ &= \frac{\frac{5}{8}\frac{2}{3}}{\frac{2}{3}} \\ &= \frac{5}{8} \end{aligned}$$

Note that $P(E2W|P1W) = P(P2W|E1W)$ because it makes no difference who is actually doing the buying. And we know from part a), that $P(E2W|P1W) = \frac{5}{8}$. So, $P(P2W|E1W) = \frac{5}{8}$.
Thus, $P(E1W|P2W) = \frac{5}{8} = P(P2W|E1W)$, so order does not matter.

3. (a) Let A_n be the event "*The gate is open at time n*". Then one has:

$$\begin{aligned} P_n &= P(A_n) \\ &= P(A_n|A_{n-1})P(A_{n-1}) + P(A_n|A_{n-1}^c)P(A_{n-1}^c) \\ &= (1-b)P_{n-1} + a(1-P_{n-1}) \\ &= (1-b-a)P_{n-1} + a. \end{aligned}$$

To solve this recurrence relation in terms of a, b, P_0 put $1 - a - b = \beta$, so that:

$$\begin{aligned} P_1 &= \beta P_0 + a, \\ P_2 &= \beta P_1 + a, \\ &= \beta^2 P_0 + \beta a + a \\ P_3 &= \beta P_2 + a \\ &= \beta^3 P_0 + a\beta^2 + a\beta + a. \end{aligned}$$

At this point one can guess that the solution is:

$$P_n = \beta^n + a \sum_{k=0}^{n-1} \beta^k. \tag{1}$$

We can prove that our guess is correct by induction. Assume that (1) is correct for n, then

$$\begin{aligned} P_{n+1} &= \beta P_n + a \\ &= \beta \left(\beta^n + a \sum_{k=0}^{n-1} \beta^k \right) + a \\ &= \beta^{n+1} + a \sum_{k=0}^{n} \beta^k, \end{aligned}$$

i.e., (1) is correct also for $n + 1$. In terms of a, b, P_0, we have found that

$$P_n = (1 - b - a)^n P_0 + a \sum_{k=0}^{n-1} (1 - b - a)^k.$$

(b) Since $|1 - b - a| < 1$ (assuming that a and b are not both 1), we have that

$$\lim_{n\to\infty} P_n = 0 + \frac{a}{1 - (1 - b - a)} = \frac{a}{a + b}.$$

The limiting probability does not depend on P_0 because the state at time n depends only on the state at time $n - 1$, and so after a sufficiently long time, the initial state will have practically no influence.

(c) The recurrence relation (1) is still valid, and we can obtain the following values for P_n.
For $a = b = 0$:

$$P_n = P_0;$$

the gate remains forever in its initial state.
For $a = 1 - b = 0$:

$$P_n = \begin{cases} P_0 & \text{for } n = 0 \\ 0 & \text{for } n > 0; \end{cases}$$

after 1 millisecond the gate becames open, no matter how it was before, and remains open forever.
For $1-a=b=0$:

$$P_n = \begin{cases} P_0 & \text{for } n=0 \\ 1 & \text{for } n>0; \end{cases}$$

after 1 millisecond the gate becames closed, no matter how it was before, and remains closed forever.
For $a=b=1$:

$$P_n = \begin{cases} P_0 & \text{for } n \text{ even} \\ 1-P_0 & \text{for } n \text{ odd}; \end{cases}$$

the gate switches every milliseconds between the two states.

4. See the answer in the textbook.

5. *(Optional Exercise).* For the "<" and ">" cases we can use the results of Problem 2. Let A be the event "*The Engineering router works*" and B the event "*The Physics router works*". Then $P(A) = \frac{6}{9} = 0.67$. On the other hand,

$$P(A|B) = \frac{5}{8} = 0.63$$

and

$$P(A|B^c) = \frac{6}{8} = 0.75.$$

For the "=" case, consider two tosses of a fair coin, with A "*Head in the first toss*" and B "*Head in the second toss*". Then

$$P(A) = \frac{1}{2}$$

and

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{1/2} = \frac{1}{2}.$$