STA 113: Probability and Statistics in Engineering
Homework # 3: Solutions

0.1 Problem 1

Let X denote the time in minutes needed to complete one component, and let Z denote a standard (mean zero, variance one) normal random variable; then

a. $P[X < 90] = P\left[\frac{X-\mu}{\sigma} < \frac{90-120}{17}\right] = P[Z < -1.764706] \approx 0.5 - (0.4608 + 0.4616)/2 = .5 - .4612 = .0388$

b. $P[X > 165] = P\left[\frac{X-\mu}{\sigma} > \frac{165-120}{17}\right] = P[Z > 2.647059] \approx 0.5 - .4960 = 0.0040$

0.2 Problem 2

Let X denote the sum, with pdf $f(x) = \begin{cases} x & 0 < x \leq 1 \\ 2-x & 1 < x \leq 2 \\ 0 & \text{elsewhere.} \end{cases}$

Then,

a. $P[X < 1/2] = \int_{-\infty}^{1/2} f(x)dx = \int_0^{1/2} x dx = (1/2)^2/2 = 1/8$

b. $E[X] = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 x^2 dx + \int_1^2 x(2-x)dx$
$= (x^3/3) \mid_0^1 + (x^2 - x^3/3) \mid_1^2 = (1/3 - 0) + (4/3 - 2/3) = 1$

0.3 Problem 3

Let X denote the spark plug lifetime, with pdf $f(x) = \begin{cases} e^{-x/10}/10 & x > 0 \\ 0 & x \leq 0 \end{cases}.$

Then:

a. $P[X > 10] = \int_{10}^{\infty} f(x)dx = \int_{10}^{\infty} \frac{e^{-x/10}}{10}dx = (-e^{-x/10}) \mid_{10}^{\infty} = e^{-1} \approx 0.3678794$

b. The probability that any *one* spark plug lasts the entire year is $\int_1^{\infty} \frac{e^{-x/10}}{10}dx = e^{-0.10}$; under independence, the probability that all six last that long is $P[(X_1 > 1) \cap (X_2 > 1) \cap \ldots \cap (X_6 > 1)] = e^{(-0.10)^6} = e^{-0.60} \approx 0.5488116$

0.4 Problem 4

Let Y denote the time in minutes the train is late, with pdf

$$f(y) = \begin{cases} \frac{3}{500}(25 - y^2) & -5 < y < 5 \\ 0 & \text{elsewhere,} \end{cases}$$

and let X=Y/60 and Z=60Y be the time in hours and seconds, respectively. Then,

a. $E[Y] = \int_{-\infty}^{\infty} y f(y) dy = \int_{-5}^{5} \frac{3}{500}(25y - y^3) dy = \frac{3}{500}(25y^2/2 - y^4/4) \mid_{-5}^{5} = 0$ min

$V[Y] = \int_{-\infty}^{\infty} (y - E[Y])^2 f(y) dy = \int_{-5}^{5} \frac{3}{500}(25y^2 - y^4) dy = \frac{3}{500}(25y^3/3 - y^5/5) \mid_{-5}^{5} = 5$ min^2

b. $E[X] = E[Y/60] = E[Y]/60 = 0$ hr

$V[X] = E[(X - E[X])^2] = E[X^2] = E[(Y/60)^2] = E[Y^2]/3600 = 5/3600 = 1/720$ hr^2

c. $E[Z] = E[Y * 60] = E[Y] * 60 = 0$ sec

$V[Z] = E[(Z - E[Z])^2] = E[Z^2] = E[(Y * 60)^2] = E[Y^2] * 3600 = 5 * 3600 = 18000$ sec^2

0.5 Optional Problem

Denote the four times by X_1, X_2, X_3, and X_4 ; then each X_j has a normal distribution with the mean $\mu_j = 120$ min and standard deviation $\sigma_j = 17$ min, while their sum $Y = X_1 + X_2 + X_3 + X_4$ has a normal distribution with mean $\mu = \mu_1 + \mu_2 + \mu_3 + \mu_4 = 480$ min and **variance** $\sigma^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 = 1156$ min^2 and hence standard deviation $\sigma = \sqrt{1156} = 34$ min, so:

$P[Y < 420 \text{ min}] = P\left[\frac{Y-\mu}{\sigma} < \frac{420-480}{34}\right] = P[Z < -1.764706] \approx 0.0388$, same as in problem 1.