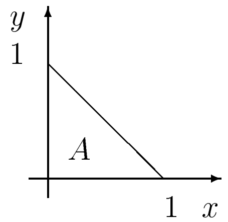
# STA113: Solutions for Homework 4

1.




(a) A density function must integrate to 1:

$$1 = \int_A f(x,y)\,dxdy = \int_A c\,dxdy = c \cdot \frac{1}{2}.$$

Hence $c = 2$.

(b) For $x$ in (0,1):

$$f_X(x) = \int f(x,y)\,dy = \int_0^{1-x} 2\,dy = 2\,(1-x).$$

(c) Let's find first the c.d.f. of $Z$. For $z$ in (0,1):

$$P(Z \leq z) = \int_{\{x+y \leq z\}} 2\,dydx = 2\int_0^z \int_0^{z-x} dydx = 2\int_0^z (z-x)\,dx = z^2.$$

Therefore

$$f_Z(z) = \frac{d}{dz}P(Z \leq z) = \begin{cases} 2z & \text{if } z \in (0,1) \\ 0 & \text{otherwise.} \end{cases}$$

(d) No, they are not independent. Since the support of $f(x,y)$, (i.e. the region where $f(x,y)$ is positive,) is not a rectangle, it cannot be that $f(x,y) = f_X(x)f_Y(y)$. It would be unwise to model these random variable as independent, since the amount of time spent on one activity cannot be spent on the other.

2. (a) joint probability mass distribution: p(x,y)

| | | X | | | |
|---|---|---|---|---|---|
| | | 0 | 1 | 2 | 3 |
| | 0 | 0 | 0 | $\frac{126}{1320}$ | $\frac{210}{1320}$ |
| Y | 1 | 0 | $\frac{126}{1320}$ | $\frac{420}{1320}$ | $\frac{126}{1320}$ |
| | 2 | $\frac{18}{1320}$ | $\frac{144}{1320}$ | $\frac{126}{1320}$ | 0 |
| | 3 | $\frac{6}{1320}$ | $\frac{18}{1320}$ | 0 | 0 |

Draw a random sample of n = 3 bridges
x = number of Duke Bridges
y = number of Failed Bridges
D = Duke, N = NCSU, F = Fail, O = OK

example: 2DF,1NO = (2 Failed Duke bridges & 1 OK NCSU bridge)
$p(x=0, y=0) = p(3NO) = 0$
$p(x=0, y=1) = p(2NO, 1NF) = 0$
$p(x=0, y=2) = p(1NO, 2NF) = 3 \cdot \left(\frac{1}{12}\frac{3}{11}\frac{2}{10}\right) = \frac{18}{1320}$
$p(x=0, y=3) = p(3NF) = \frac{3}{12}\frac{2}{11}\frac{1}{10} = \frac{6}{1320}$

$$
\begin{aligned}
p(x=1,y=0) &= p(1DO,2NO) \\
&= 0 \\
p(x=1,y=1) &= p(1DF,2NO)+p(1DO,1NO,1NF) \\
&= 0+3!\cdot\frac{7}{12}\frac{1}{11}\frac{3}{10} \\
&= \frac{126}{1320} \\
p(x=1,y=2) &= p(1DF,1NF,1NO)+p(1DO,2NF) \\
&= 3!\cdot\frac{1}{12}\frac{3}{11}\frac{1}{10}+\binom{3}{1}\cdot\frac{7}{12}\frac{3}{11}\frac{2}{10} \\
&= \frac{144}{1320} \\
p(x=1,y=3) &= p(1DF,2NF) \\
&= \binom{3}{1}\cdot\frac{1}{12}\frac{3}{11}\frac{2}{10} \\
&= \frac{18}{1320} \\
p(x=2,y=0) &= p(2DO,1NO) \\
&= \binom{3}{1}\cdot\frac{7}{12}\frac{6}{11}\frac{1}{10} \\
&= \frac{126}{1320} \\
p(x=2,y=1) &= p(1DF,1DO,1NO)+p(2DO,1NF) \\
&= 3!\cdot\frac{1}{12}\frac{7}{11}\frac{1}{10}+\binom{3}{1}\cdot\frac{7}{12}\frac{6}{11}\frac{3}{10} \\
&= \frac{420}{1320} \\
p(x=2,y=2) &= p(2DF,1NO)+p(1DF,1DO,1NF) \\
&= 0+3!\cdot\frac{1}{12}\frac{7}{11}\frac{3}{10} \\
&= \frac{126}{1320} \\
p(x=2,y=3) &= p(2DF,1NF) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
p(x=3, y=0) &= p(3DO) \\
&= \frac{7}{12}\frac{6}{11}\frac{5}{10} \\
&= \frac{210}{1320} \\
p(x=3, y=1) &= p(2DO, 1DF) \\
&= \binom{3}{1} \cdot \frac{7}{12}\frac{6}{11}\frac{1}{10} \\
&= \frac{126}{1320} \\
p(x=3, y=2) &= p(2DF, 1DO) \\
&= 0 \\
p(x=3, y=3) &= p(3DF) \\
&= 0
\end{aligned}
$$

(b)

$$
\begin{aligned}
E(x) &= \frac{1}{1320}\left(0 \cdot 24 + 1 \cdot 288 + 2 \cdot 672 + 3 \cdot 336\right) \\
&= \frac{1}{1320}(2640) \\
&= 2 \\
E(Y) &= \frac{1}{1320}\left(0 \cdot 336 + 1 \cdot 672 + 2 \cdot 288 + 3 \cdot 24\right) \\
&= \frac{1}{1320}(1320) \\
&= 1
\end{aligned}
$$

(c)

$$\begin{aligned}
E(xy) &= \frac{1}{1320}\left(0+1(126)+2(420+144)+3(126+18)+4(126)+6(126+0)\right) \\
&= \frac{1}{1320}(126+1128+432+504) \\
&= \frac{1}{1320}(2190) \\
&= \frac{219}{132} \\
&\approx 1.6591
\end{aligned}$$

(d) No, x and y are not independent because
$p(x=1) = \frac{\binom{8}{1}\binom{4}{2}}{\binom{12}{3}} = \frac{48}{220}$
$p(y=1) = \frac{\binom{4}{1}\binom{8}{2}}{\binom{12}{3}} = \frac{12}{220}$
$p(x=1)\cdot p(y=1) = \frac{576}{48400} \approx .01190$
$p(x=1,y=1) = \frac{125}{1320} \approx .09470$
$p(x=1,y=1) \neq p(x=1)\cdot p(y=1)$

(e)

$$\begin{aligned}
Cov(xy) &= E(xy) - E(x)\cdot E(y) \\
&= \frac{2190}{1320} - 2\cdot 1 \\
&= -\frac{45}{132} \\
&\approx -0.3409
\end{aligned}$$

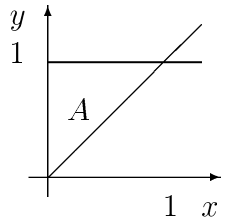
As the number of Duke bridges increase, the number of Failed bridges decrease.

3. (a) We can reasonably assume that the failure time, $X$, of a plug is independent of the time, $Y$, you intended to replace it. In this case the joint p.d.f. of $X$ and $Y$ is simply the product of the two marginal p.d.f.'s:

$$f(x,y) = f_X(x)\cdot f_Y(y) = \begin{cases} 0.1e^{-0.1x} & \text{if } x>0 \text{ and } 0<y<1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, if $A = \{x < y\}$ (see picture)

$$
\begin{aligned}
P(X < Y) &= \int_A f(x, y)\, dxdy \\
&= \int_0^1 \int_0^y 0.1e^{-0.1x}\, dxdy \\
&= \int_0^1 (1 - e^{-0.1y})\, dy \\
&= \Big[y + 10e^{-0.1y}\Big]_0^1 \\
&= 1 + 10e^{-0.1} - 10 = 0.04837.
\end{aligned}
$$




(b) Let $X_1, ..., X_6$ be the failure times of the six spark-plugs and $Z$ be the time when the first failure of a spark-plug occurs. Observe that $Z = \min_i X_i$. To compute the c.d.f. of $Z$ it is easier to think in terms of the survival function $P(Z > z)$:

$$
\begin{aligned}
P(Z > z) &= P(\bigcap_{i=1}^{6}\{X_i > z\}) = \prod_{i=1}^{6} P(X_i > z) \\
&= P(X_1 > z)^6 = e^{-0.6z}.
\end{aligned}
$$

Therefore

$$F_Z(z) = P(Z \le z) = 1 - e^{-0.6z}.$$

The time $T$ must satisfy the following inequality:

$$F_Z(T) \le 0.01,$$

and solving the inquality we find

$$\begin{aligned} 1 - e^{-0.6T} &\leq 0.01 \\ e^{-0.6T} &\geq 0.99 \\ -0.6T &\geq \log 0.99 \\ T &\leq 0.01675. \end{aligned}$$

So, $T$ cannot be greater than 0.01675 years, approximately 6 days.

4. $l = \frac{1}{2}y_1 - y_2 + 2y_3$

$$\vec{a} = \begin{bmatrix} \frac{1}{2} \\ -1 \\ 2 \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 3 & -2 \\ 4 & -2 & 9 \end{bmatrix}$$

$$\begin{aligned} mean &= E[l] \\ &= \vec{a}^T \vec{u} \\ &= \begin{bmatrix} \frac{1}{2} - 12 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 \\ 1 & 3 & -2 \\ 4 & -2 & 9 \end{bmatrix} \\ &= 11 \\ \sigma_l^2 &= V[l] \\ &= \vec{a}^T C \vec{a} \\ &= \begin{bmatrix} \frac{1}{2} - 12 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 \\ 1 & 3 & -2 \\ 4 & -2 & 9 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -1 \\ 2 \end{bmatrix} \\ &= 54.5 \end{aligned}$$

5. (*Optional Exercise.*) Let $F = X/(X+Y)$. For any real number $s$ we have:

$$\begin{aligned} F \le s \quad &\text{iff} \quad \frac{X}{X+Y} \le s \\ &\text{iff} \quad X \le sX + sY \\ &\text{iff} \quad sY \ge (1-s)X \\ &\text{iff} \quad Y \ge \frac{1-s}{s} X. \end{aligned}$$

(where iff is a shorthand notation for "if and only if".) Therefore, taking $s$ in the interval (0,1), we obtain (see picture):

$$P(F \le s) = P\left(Y \ge \frac{1-s}{s} X\right) = 2 \cdot \frac{1 \cdot s}{2} = s.$$

Deriving with respect to $s$ we can easily deduce that $F$ has a uniform distribution on (0,1), with p.d.f. given by

$$f(s) = \begin{cases} 1 & \text{if } 0 < s < 1 \\ 0 & \text{otherwise.} \end{cases}$$