STA 113: Probability and Statistics in Engineering
Homework # 7: Solutions

0.1 Problem 1

a.) $\begin{array}{l} H_0: \ \ \lambda = 1 \\ H_a: \ \ \lambda = 3 \end{array}$

The p-value is the probability of observing a test statistic at least as contradictory of the null hypothesis as the value of the test statistic actually observed.

What test statistic is appropriate here? The sum of the observations seems like the best choice. This is because the sum of Poisson random variables is also a Poisson random variable with mean equal to the sum of the means of the random variables. In this case, under the null hypothesis, $X_i \sim Poi(1)$, i.e., each observation is Poisson distributed with mean of 1. Therefore $\sum_{i=1}^{4} X_i \sim Poi(1 + 1 + 1 + 1)$, i.e., the sum of the observations is Poisson distributed with mean of 4. The p-value is thus the probability of observing a sum of 15 or larger, as values larger than 15 are even more contradictory of the null hypothesis than the value of 15.

Let the test statistic $S = \sum_{i=1}^{4} X_i$. What is $P[S \geq 15 \mid \lambda_S = 4] =??$

Use the following minitab code to find the answer:

```
cdf 14;
poisson 4.
```

Minitab gives $P[Y \leq 14 \mid \lambda_S = 4] = 1$; $1 - 1 = 0$, which is the p-value for this hypothesis test. In other words, the probability of getting observations at least as contradictory or more contradictory than the null hypothesis is very, very small, effectively zero. S-plus gives a more exact answer that $p = .00002$.

b.) This requires use of Bayes' rule with, $P[\lambda_S = 4] = 0.5, P[\lambda_S = 12] = 0.5$. Since we're using the sum of the observations as our statistic, we need to remember to multiply each original lambda by 4.

Bayes' rule is:

$$P[B \mid A] = \frac{P[A \mid B] * P[B]}{(P[A \mid B] * P[B]) + (P[A \mid B^C] * P[B^C])}$$

$$B : \lambda_S = 4, \ B^C : \lambda_S = 12, \ A : S = 15$$

Plugging the numbers in:

$$P[\lambda_S = 4 \mid S = 15] = \frac{(P[S = 15 \mid \lambda_S = 4] * P[\lambda_S = 4])}{(P[S = 15 \mid \lambda_S = 4] * P[\lambda_S = 4)] + (P[S = 15 \mid \lambda_S = 12] * P[\lambda_S = 12])}$$

To calculate this, use the probability mass function for the Poisson distribution as appropriate. For example,

$$P[S = 15 \mid \lambda_S = 4] = \frac{e^{-\lambda} * \lambda^S}{S!} = \frac{e^{-4} * 4^{15}}{15!} = 1.5X10^{-5}$$

When the smoke clears:

$$P[\lambda_S = 4 \mid S = 15] = .0002$$

The conclusion is that the probability of the data being from off-peak is very small. The probability of the data being from peak would be $1 - .0002 = .9998$, and we would conclude that the data are from peak period.

Note that using the approach in part a.) with different data might lead you to conclude in favor of the null hypothesis (not reject the null hypothesis) even if the alternative hypothesis is more likely given the data.

0.2 Problem 2

$\overline{X}_A = 78, \quad s_A = 8, \quad n_A = 10$
$\overline{X}_B = 85, \quad s_B = 10, \quad n_B = 6$

a.) There is no indication that the scores are paired in any way. Given that there are more scores for A than for B, the scores must be considered to be independent samples.

b.) Use small sample methods. Both $n_A < 30$ and $n_B < 30$. Small sample methods assume normal data and equal variance for the two populations. If you use the modification in the text for unequal variance, that's fine, too. Generally, statisticians start off with a null hypothesis that variances are equal and then test this hypothesis. If they don't reject the null hypothesis, they then perform tests assuming equal variance. Note that the sample variances of two samples will almost always be at least a little different - this does not necessarily lead one to conclude that the underlying population variances are different.

c.) $\begin{array}{ll} H_0: & \mu_A - \mu_B = 0 \\ H_a: & \mu_A - \mu_B \neq 0 \end{array}$

I use an alternative of not equal (i.e., a two-sided test) because the question asks whether the machines have *identical* fitness. Under the null hypothesis, $t = \frac{\overline{X}_A - \overline{X}_B}{\sqrt{s_p^2 * \left(\frac{1}{n_A} + \frac{1}{n_B}\right)}}$, if we assume that the underlying population variances of the two populations are equal.

$$s_p^2 = \frac{(n_A - 1) * s_A^2 + (n_B - 1) * s_B^2}{n_A + n_B - 2}$$

Since the two sample variances are not all that different, I have done the test assuming equal population variances. You can also use the test procedure outlined in the book for cases when the underlying population variances are assumed to be different, in which case the number of degrees of freedom is adjusted.

The statistic t is distributed according to the t distribution, with $n_A - n_B - 2$ degrees of freedom, if the null hypothesis is true.

$$t = \frac{78 - 85}{\sqrt{\frac{(9*8^2+5*10^2)}{10+6-2} * \left(\frac{1}{10} + \frac{1}{6}\right)}} = -1.546$$

Using the t distribution with 14 df, $P[T < -1.546] + P[T > 1.546] = 0.0722$.

This can be obtained in minitab with the following code:

```
cdf -1.546;
t 14.
```

Twice the resulting value $2 * .0722 = .144$, the p-value for the two-sided test. Using the table in the back of the text only allows you to see that $p > 0.10$.

The evidence does not support rejecting the null hypothesis. The probability of observing these data or more extreme data given the null hypothesis is very high. There is not enough evidence to reject the null hypothesis.

d.) $\sigma_A^2 = \sigma_B^2 = 100$. Now we can use the following test statistic:

$$z = \frac{\overline{X}_A - \overline{X}_B}{\sigma_{\overline{X}_A - \overline{X}_B}} \text{ where } \sigma_{\overline{X}_A - \overline{X}_B} = \sqrt{\frac{\sigma_A^2}{n_A} + \frac{\sigma_B^2}{n_B}}$$

because $\sigma^2_{\overline{X}_A - \overline{X}_B} = \sigma^2_{\overline{X}_A} + \sigma^2_{\overline{X}_B} - 2 * cov(\overline{X}_A, \overline{X}_B) = \frac{\sigma_A^2}{n_A} + \frac{\sigma_B^2}{n_B} - 0$

Under the null hypothesis, $z \sim N(0, 1)$.

$$z = \frac{78 - 85}{\sqrt{\frac{100}{10} + \frac{100}{6}}} = -1.36.$$

Again, the p-value is the probability of finding evidence at least as contradictory of the null hypothesis as the evidence observed, therefore for the two-sided test, we find the probability that Z is more extreme than the value observed.

$$P[Z < -1.36] + P[Z > 1.36] = 2 * (0.5 - 0.4131) = 0.174.$$

We still fail to reject the null hypothesis, but when given the value for σ_A and σ_B the p-value is .174 rather than .144.

0.3 Problem 9.14

Since EPA is interested in testing if the amount *exceeds* the allowable amount, use a one-sided alternative.

$$\begin{aligned} H_0 &: \mu_{radium} = 4 \\ H_a &: \mu_{radium} > 4 \end{aligned}$$

0.4 Problem 9.15

Since the engineers are interested in whether the methods *differ*, rather than whether a specific method produces more than the other method, use a two-sided alternative.

$$\begin{array}{ll} H_0: & \mu_1 - \mu 2 = 0 \\ H_a: & \mu_1 - \mu 2 \neq 0 \end{array}$$

0.5 Problem 9.18a

$\overline{y} = 0.044, s_y = 0.884, n = 197.$

$$\begin{array}{ll} H_0: & \mu = 0 \\ H_a: & \mu > 0 \end{array}$$

We use a one-sided alternative because the question asks if the inbreeding coefficient *exceeds* 0.

Given the large number of samples, we can use z as our test statistic: $z = \frac{\overline{y}}{\sigma_{\overline{y}}}$, with $\sigma_{\overline{y}} = \frac{\sigma_y}{\sqrt{n}} \approx \frac{s_y}{\sqrt{n}}$. Therefore, $z = \frac{0.044}{\frac{0.884}{\sqrt{197}}} = 0.699$. The probability of evidence at least as contradictory as that observed is: $P[Z > 0.699] \approx 0.5 - 0.258 \approx 0.242$. We cannot reject the null hypothesis at $\alpha = .05$.

0.6 Problem 9.28

Since the question asks for two-sided tests, calculate $P[Z < -\mid z \mid] + P[Z > \mid z \mid]$ where z (lower-case) is the value specified for in the question. For example for part a.) any values of $Z < -1.01$ or $Z > 1.01$ are at least as contradictory of the null hypothesis as the value $z = -1.01$ that is actually observed because the values are at least as far from zero as -1.01.

a.) $2 * (0.5 - 0.3438) = 0.3124$

b.) $2 * (0.5 - 0.4911) = 0.0178$

c.) $2 * (0.5 - 0.5) = 0$

d.) $2 * (0.5 - 0.4265) = 0.147$

0.7 Problem 9.34

a.) Since different individuals are used in the groups, this appears to be a case of independent samples. The hypothesis is that the mean times do not differ between the two displays, so an appropriate test would be a test of the following null hypothesis: $\begin{array}{ll} H_0: & \mu_1 - \mu 2 = 0 \\ H_a: & \mu_1 - \mu 2 \neq 0 \end{array}$

We would use a small sample test, since both sample sizes are less than 30.

The small sample test uses the test statistic $t = \frac{\overline{y}_1 - \overline{y}_2}{\sqrt{s_p^2 * \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$ where t is distributed according to the t distribution with $n_1 + n_2 - 2$ degrees of freedom. Refer back to problem 2c for the formula for s_p^2.

b.) The assumptions required for this test are that the populations of times are approximately normal, that the variances of the populations of times are equal, and that the time of each person is independent of the times of the other people.

c.) We also need the standard deviations for the two groups in order to compute t.

d.) If $p > .10$ this indicates that the probability of data that contradict the null hypothesis at least as strongly as the data observed is greater than 0.10. If we use a significance level of 0.10 or smaller, there is not enough evidence to reject the null hypothesis. The data are not strong enough to contradict the null hypothesis such that the chance of rejecting the null hypothesis when it is actually correct is a very small value (such as .05 or .10)

0.8 Problem 9.35

$$H_0: \quad \mu_{fog} - \mu_{cl/cl} = 0$$
$$H_a: \quad \mu_{fog} - \mu_{cl/cl} \neq 0$$

In other words, we want to test the hypothesis that the mean oxon/thion ratio is the same under fog and clear/cloudy conditions. Given the small sample sizes, we have to assume normal data and equal variances and compute the statistic:

$$t = \frac{\overline{y}_{fog} - \overline{y}_{cl/cl}}{\sqrt{s_p^2 * \left(\frac{1}{n_{fog}} + \frac{1}{n_{cl/cl}}\right)}}$$

Under the null hypothesis, t should be distributed according to the t distribution with $n_{fog} - n_{cl/cl} - 2$ degrees of freedom.

Refer back to problem 2c to calculate s_p^2.

Remember the sample standard deviation: $s^2 = \sqrt{\frac{\sum_{i=1}^{n}(y_i - \overline{y})^2}{n-1}}$, which is plugged into the formula for s_p^2.

We leave out the data from the fog on Jan 20, because of the lost oxon sample.

I calculate: $\overline{y}_{fog} = 0.274, s_{fog} = .119, \overline{y}_{cl/cl} = 0.452, s_{cl/cl} = 0.186$. Using these values, $t = -2.05$. Since this is a two-sided test, $p = P[T < -2.05] + P[T > 2.05] = 0.0679$. I calculated this p-value on the computer, but the table in the text tells you that with 10 df, the p-value for a one-sided test is between .025 and .05, therefore, the p-value for the two-sided test is twice this value, i.e., between .05 and .10. We would reject the null hypothesis at the .10 level, but not at the .05 level.

0.9 Problem 9.72

This question asks about the precision of two instruments - if one instrument is more precise than another, we would expect the variance of measurements from this instrument to be less than the variance from the other instrument. Therefore, we want to test:

$$\begin{array}{ll} H_0: & \sigma_2^2 = \sigma_1^2 \\ H_a: & \sigma_2^2 > \sigma_1^2 \end{array}$$

We consider the samples to be independent samples, since this is a test of variances, not a test of means, and, in any event there is no indication of paired samples.

$\frac{s_1^2 * (n_1 - 1)}{\sigma_1^2} \sim \chi^2$ with $n_1 - 1$ degrees of freedom. $\frac{s_2^2 * (n_2 - 1)}{\sigma_2^2} \sim \chi^2$ with $n_2 - 1$ degrees of freedom.

The ratio of two χ^2 random variables is distributed according to the F distribution, with $n_1 - 1$ and $n_2 - 1$ degrees of freedom. Under the null hypothesis, $\sigma_1^2 = \sigma_2^2$. Also, for these data, $n_1 - 1 = n_2 - 1$. Therefore, if the null hypothesis is true, the ratio of these χ^2 random variables simplifies to

$$\frac{s_2^2}{s_1^2} \sim F_{4,4} \text{ where } s_1^2 = 1, s_2^2 = 10$$

The p-value is the probability of finding a value of the test statistic at least as contradictory as the value observed, i.e.,

$$p = P[F_{4,4} > 10] = .0233. \text{ From the text, one can only determine } p < .025$$

There is sufficient evidence to reject the null hypothesis of equal variance at significance levels of .10, .05, and .025. We can conclude that instrument 1 is more precise (has smaller variance) than instrument 2.

0.10 Problem 9.74

This problem can be solved using a small sample test for difference of means with paired samples, since there is an experimental result and a theoretical result for each of 6 T-beams.

Since the questions asks if there is a difference, rather than whether one exceeds the other, we use a two-sided alternative:

$$\begin{array}{ll} H_0: & \mu_E - \mu_T = 0 \\ H_a: & \mu_E - \mu_T \neq 0 \end{array}$$

The vector of differences between experimental and theoretical is:

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(0.07  0.55 -0.20 -0.20  0.68  1.18)
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We use the test statistic: $t = \frac{\overline{d}}{s_d/\sqrt{n}}$ where $\overline{d} = 0.347$ is the mean of the differences, $s_d = 0.552$ is the standard deviation of the differences, and $n = 6$ is the number of differences (number of pairs). Under the null hypothesis, t is distributed according to the t distribution with $n - 1 = 5$ degrees of freedom.

$t = 1.54$, so with a two-sided test, $p = P[T < -1.54] + P[T > 1.54] = .184$. I calculated this by computer - the best you can do using the table in the text is to find that for a one-sided test, $.05 < p < .10$ which means that for the two-sided test, the p-value is twice as much: $.10 < p < .20$. There is not enough evidence to reject the null hypothesis at any level of significance equal to or less than .10.