Midterm 1 – Solutions

1. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $y = (y_1, \ldots, y_n)$. There are several ways to compute $p(\beta|y)$. For example,

$$p(\beta|y) \propto p(\beta)\, \frac{p(\lambda|\beta)\, p(y|\beta,\lambda)}{p(\lambda|\beta,y)} = p(\beta) \prod_{i=1}^{n} \frac{p(\lambda_i|\beta)\, p(y_i|\beta,\lambda_i)}{p(\lambda_i|\beta,y_i)} \tag{1}$$

To evaluate (1) use

$$p(\lambda|y,\beta) \propto \prod_{i=1}^{n} (\beta+1)^{\alpha+y_i} \lambda_i^{\alpha+y_i-1} e^{-\lambda_i(\beta+1)}$$

and find

$$p(\beta|y) \propto \beta^{c-1} e^{-d\beta} \prod_{i=1}^{n} \frac{\beta^{\alpha} \lambda_i^{\alpha-1} e^{-\beta\lambda_i} e^{-\lambda_i} \lambda_i^{y_i}}{(\beta+1)^{\alpha+y_i} \lambda_i^{\alpha+y_i-1} e^{-\lambda_i(\beta+1)}} = \frac{\beta^{c+n\alpha-1} e^{-d\beta}}{(1+\beta)^{n\alpha+n\bar{y}}}$$

2a. Let $a = 10, b = 20, c = 20, d = 10$ denote the prior parameters. Let $a^* = a + x$, $b^* = b + n - x$, $c^* = c + x$ and $d^* = d + n - x$.

$$p(\theta|x) \propto p(\theta)p(x|\theta) = \underbrace{\frac{1}{2}\frac{B(a^*,b^*)}{B(a,b)}}_{\pi_1} \underbrace{\frac{\theta^{a^*-1}(1-\theta)^{b^*-1}}{B(a^*,b^*)}}_{Be(a^*,b^*)} + \underbrace{\frac{1}{2}\frac{B(c^*,d^*)}{B(c,d)}}_{\pi_2} \underbrace{\frac{\theta^{c^*-1}(1-\theta)^{d^*-1}}{B(c^*,d^*)}}_{Be(c^*,d^*)} \tag{2}$$

2b. Note $\int \theta Be(a,b)d\theta = a/(a+b)$ (expected value of a $Be(a,b)$ r.v. Thus

$$E(\theta|x) = \int x p(\theta|x)dx = \frac{\pi_1}{\pi_1+\pi_2} \cdot \frac{a^*}{a^*+b^*} + \frac{\pi_2}{\pi_1+\pi_2} \cdot \frac{c^*}{c^*+d^*}.$$

Don't forget to standardize the posterior by dividing by $(\pi_1 + \pi_2)$. The expression in (2) is not yet standardized [1].

3a. Let $S = \sum x_i$. Then $p(\theta|x) \propto \theta^{\alpha+n-1} e^{-(\beta+S)\theta}$, i.e. $Ga(\alpha+n, \beta+S)$.

3b. Let $\tilde{x} = x_{n+1}$, and $S = \sum x_i$. Then

$$\begin{aligned} p(\tilde{x}|x) &= \int p(\tilde{x}|\theta)p(\theta|x)d\theta = \int \theta^{\alpha+n+1-1} e^{-(\beta+S+\tilde{x})\theta} d\theta \\ &\propto [1 + \tilde{x}/\beta']^{-(\alpha+n+1)}, \end{aligned}$$

where $\beta' = \beta + \sum x_i$.

3c Without loss of generality assume that observations $i = 1, \ldots, m$ are uncensored (i.e. the failure time is observed), and observations $i = m+1, \ldots, n$ are censored at $x_i = a$. Note $Pr(x_i > a|\theta) = exp(-\beta a)$ and

$$p(x|\theta) = \prod_{i=1}^{m} \theta e^{-\theta x_i} \prod_{i=m+1}^{n} e^{-\theta a} = \theta^m e^{-\theta\left(\sum_{i=1}^{m} x_i + (n-m)a\right)}.$$

Hence $p(\theta|x) = Ga[\alpha+m, \beta+\sum_{i=1}^{m} x_i + (n-m)a]$ and $p(\tilde{x}|x) \propto [1+\tilde{x}/\beta'']^{-(\alpha+m+1)}$, where $\beta'' = \beta + \sum_{i=1}^{m} x_i + (n-m)a$.

[1] I did forget – thanks Daniel for pointing out!