Homework 9a – Solutions

8.8 a. As y_i iid $\sim Bernoulli(p)$, then $E[y_i] = \mu = p$. Setting this expression to its respective sample moment, we obtain: $\hat{\mu} = \frac{\sum_{i=1}^{n} y_i}{n}$, or $\hat{p} = \frac{y}{n}$

8.8 c. For the Binomial experiment, the likelihood function is $L = K \times p^y \times (1-p)^{n-y}$ where $K = \frac{n!}{y! \times (n-y)!}$ is a constant which doesn't depend on p. The log-likelihood is: $ln(L) = ln(K) + y \times p + (n-y) \times (1-p)$. Taking the first derivative of ln(L) with respect to p and setting it equal to zero yields:

$$\frac{dln(L)}{dp} = 0 \Rightarrow \frac{y}{\hat{p}} - \frac{(n-y)}{1-\hat{p}} = 0$$

And so, $\hat{p} = y/n$

8.9 a. Since $y_1, \cdots, y_n$ are independent, it follows that the likelihood function is:

$$L = p(y_1) \times p(y_2) \times \cdots p(y_n) = \prod_{i=1}^{n} p(y_i) = \prod_{i=1}^{n} \frac{e^{-\lambda} \times \lambda_i^y}{y_i!} = \frac{e^{-\lambda \times n} \times \lambda^{\sum_{i=1}^{n} y_i}}{\prod_{i=1}^{n} y_i!}.$$

The log-likelihood is: $ln(L) = -n \times \lambda + \sum_{i=1}^{n} y_i \times ln(\lambda) + K$, where K is a constant which does not depend on λ.

By taking the first derivative of ln(L) with respect to λ and setting it equal to zero we obtain:

$$\frac{dln(L)}{d\lambda} = 0 \Rightarrow -n + \frac{\sum_{i=1}^{n} y_i}{\hat{\lambda}} = 0$$

So, $\hat{\lambda} = \frac{\sum_{i=1}^{n} y_i}{n}$

8.17. By Theorem 7.2, the sampling distribution of $\bar{y}$ is approximately normal with mean $\mu_{\bar{y}} = \mu = \lambda$ and standard deviation $\sigma_{\bar{y}} = \sigma/\sqrt{n} = \sqrt{\lambda/n}$. Thus, $z = \frac{\bar{y}-\lambda}{\sqrt{\lambda/n}}$ has an approximate standard normal distribution. Using z as the pivotal statistic, the confidence interval for λ is:

$$P(-z_{\alpha/2} \leq z \leq z_{\alpha/2}) = P(-z_{\alpha/2} \leq \frac{\bar{y}-\lambda}{\sqrt{\lambda/n}} \leq z_{\alpha/2}) = 1-\alpha.$$

We can estimate the standard deviation by using $\bar{y}$ which is the maximum likelihood estimate of λ (Exercise 8.9). Then we obtain:

$$P(-z_{\alpha/2} \leq \frac{\bar{y}-\lambda}{\sqrt{\bar{y}/n}} \leq z_{\alpha/2}) = 1-\alpha$$

$$\Rightarrow P(-z_{\alpha/2} \times \sqrt{\bar{y}/n} \leq \bar{y} - \lambda \leq z_{\alpha/2} \times \sqrt{\bar{y}/n}) = 1-\alpha$$

$$\Rightarrow P(-\bar{y} - z_{\alpha/2} \times \sqrt{\bar{y}/n} \leq -\lambda \leq -\bar{y} + z_{\alpha/2} \times \sqrt{\bar{y}/n}) = 1-\alpha$$

$$\Rightarrow P(\bar{y} - z_{\alpha/2} \times \sqrt{\bar{y}/n} \leq \lambda \leq \bar{y} + z_{\alpha/2} \times \sqrt{\bar{y}/n}) = 1-\alpha$$

So, the interval with $(1-\alpha)\%$ confidence for λ is :

$$[\bar{y} - z_{\alpha/2} \times \sqrt{\bar{y}/n}, \bar{y} + z_{\alpha/2} \times \sqrt{\bar{y}/n}]$$

8.18. By Theorem 7.2, the sampling distribution of $\bar{y}$ is approximately normal with mean $\mu_{\bar{y}} = \mu = \beta$ and standard deviation $\sigma_{\bar{y}} = \sigma/\sqrt{n} = \sqrt{\beta^2/n} = \beta/\sqrt{n}$. Thus, $z = \frac{\bar{y}-\beta}{\beta/\sqrt{n}}$ has an approximate standard normal distribution.

Using z as the pivotal statistic, the confidence interval for λ is:

$$P(-z_{\alpha/2} \leq z \leq z_{\alpha/2}) = P(-z_{\alpha/2} \leq \frac{\bar{y}-\beta}{\beta/\sqrt{n}} \leq z_{\alpha/2}) = 1-\alpha.$$

We can estimate the standard deviation by using $\bar{y}$ which is the maximum likelihood estimate of β (Example 8.4). Then we obtain:

$$P(-z_{\alpha/2} \leq \frac{\bar{y}-\beta}{\bar{y}/\sqrt{n}} \leq z_{\alpha/2}) = 1-\alpha$$

$$\Rightarrow P(-z_{\alpha/2} \times \bar{y}/\sqrt{n} \leq \bar{y}-\lambda \leq z_{\alpha/2} \times \bar{y}/\sqrt{n}) = 1-\alpha$$

$$\Rightarrow P(-\bar{y} - z_{\alpha/2} \times \bar{y}/\sqrt{n} \leq -\lambda \leq -\bar{y} + z_{\alpha/2} \times \bar{y}/\sqrt{n}) = 1-\alpha$$

$$\Rightarrow P(\bar{y} - z_{\alpha/2} \times \bar{y}/\sqrt{n} \leq \lambda \leq \bar{y} + z_{\alpha/2} \times \bar{y}/\sqrt{n}) = 1-\alpha$$

Therefore, the $(1-\alpha)\%$ confidence interval for β is:

$$[\bar{y} - z_{\alpha/2} \times \bar{y}/\sqrt{n}, \bar{y} + z_{\alpha/2} \times \bar{y}/\sqrt{n}]$$

8.24 a. 0.95

8.24 b. If we were to repeatedly collect a sample of size 6 from the population of passive samplers and construct, for each sample, a $(1-\alpha)\%$ confidence interval for the mean sampling rate, then we expect $(1-\alpha)\%$ of the intervals to enclose the true value of the mean sampling rate.

8.24 c. Since the confidence coefficient is 0.95, we say that we are 95% confident that the interval from 49.66 to 51.48 contains the true mean sampling rate.

8.24 d. We have to assume that the sampling rates are normally distributed.

8.26 a. The confidence intervals can be calculated by using the following expression:

$$[\bar{y} - z_{\alpha/2} \times s/\sqrt{n}, \bar{y} + z_{\alpha/2} \times s/\sqrt{n}]$$

For intervals with 90% confidence, we have: $z_{0.10/2} = z_{0.05} = 1.64$. By applying this expression we obtain:

For Late gabbro: $[3.04 - 1.64 \times \frac{0.13}{\sqrt{36}}, 3.04 + 1.64 \times \frac{0.13}{\sqrt{36}}] = [3.004467, 3.075533]$

For Massive gabbro: $[2.83 - 1.64 \times \frac{0.11}{\sqrt{148}}, 2.83 + 1.64 \times \frac{0.11}{\sqrt{148}}] = [2.815171, 2.844829]$

For Cumberlandite: $[3.05 - 1.64 \times \frac{0.31}{\sqrt{135}}, 3.05 + 1.64 \times \frac{0.31}{\sqrt{135}}] = [3.006244, 3.093756]$

8.26 b.

Considering the late gabbro rock, are 90% confident that the interval from [3.00,3.08] contains the true mean density.

For Massive gabbro, we are 90% confident that the interval from [2.82,2.84] contains the true mean density.

Based on Cumberlandite, weare 90% confident that the interval from [3.01,3.09] contains the true mean density.