Name:

STAT 215 – Midterm 2

Time: You have 3 hours time for this take-home exam. If you are not finished after 3 hours and think that you could still do substantially more work with additional time you can continue if you clearly indicate what you did within the first 3 hours. Use a different color pen for work beyond 3 hours.

Please be honest about the time limit. Depending on the circumstances I will only take off minor points for running over time. You are on your honor.

Due Date: The exam is due back into my mailbox by

Friday April 30, 5pm.

If you need more time and you tell me ahead of time you can submit the exam by

Wednesday May 5, 5pm (EDT) *by fax* to (011 562) 552-5916 (Chile).

If you plan to do so, please let me know ahead of time so I can make appropriate arrangments.

1. Suppose $X_1, \ldots, X_n$ be independently and identically distributed with density

$$f(x|\theta) = \frac{1}{\sigma}\exp\{-(x-\mu)/\sigma\}, \quad x \geq \mu$$

where $\theta = (\mu, \sigma^2)$, $-\infty < \mu < \infty, \sigma^2 > 0$.

(i) Find the m.l.e. estimates of μ and σ^2.

(ii) Find the m.l.e. of $\eta = P_\theta[X_1 \geq t]$ for $t > \mu$.

2. Let X be a $N(\theta, 1)$ variable and consider the estimate $T_{a,b}(X) = aX + b$ of θ.

(i) Calculate the mean squared error $R(\theta, T_{a,b})$ of $T_{a,b}$.

(ii) Plot $R(\theta, T_{1/2,0})$ and $R(\theta, T_{1,0})$ as a function of θ and show that neither estimate improves the other for all θ.

(iii) Is there any estimate of the form $T_{a,b}(X) = aX + b$ which improves on $T_{1,0} = X$ for all θ?

(iv) Show that X is the only unbiased estimate of the form $aX + b$.

3. Let $(X_1, \ldots, X_n)$ be the indicators of n binomial trials with probability of success θ, $0 \leq \theta \leq 1$.

(i) Show directly that $S = \sum_{i=1}^{n} X_i$ is complete.

Hint:

$$\sum_{k=0}^{n} \binom{n}{k} g(k)\theta^k(1-\theta)^{n-k} = (1-\theta)^n \left\{ \sum \binom{n}{k} g(k)\rho^k \right\},$$

where $\rho = \theta/(1-\theta)$. Use the fact that a power series which is identically zero in a neighborhood of 0 must have *all* coefficients ero.

(ii) Deduce that $\bar{X}$ is a U.M.V.U. estimate of θ.

4. Assume a sampling model $\underline{\mathbf{x}} \sim p_\theta(\underline{\mathbf{x}})$ parametrized by some parameter θ. Consider a hypothesis test for H_0: $\theta = \theta_0$. Assume $T(\underline{\mathbf{x}})$ is a test statistic which has a density when $\theta = \theta_0$. Let $\alpha(t)$ denote the p-value (observed significance level) for an observed $T = t$. Show that $\alpha(T)$ has a uniform $U(0, 1)$ distribution when $\theta = \theta_0$. [1]

[1] Don't be disturbed if the solution is very short and simple – it's correct.

5. Let X_i be the number of arrivals at a service counter on the ith of a sequence of n days. A possible model for this data is to assume that customers arrive according to a homogeneous Poisson process, i.e., the X_i are a sample from a Poisson distribution with parameter θ, the expected number of arrivals per day. Suppose that if $\theta \leq \theta_0$ it is not worth keeping the counter open.

Exhibit the optimal test statistic $T(\underline{\mathbf{X}})$ for H_0: $\theta \leq \theta_0$ vs. H_1: $\theta > \theta_0$. I.e., find a U.M.P. test δ with rejection region of the type $R = \{T(\underline{\mathbf{X}}) \geq c\}$.

6. Let $X_1, \ldots, X_n$ be a $N(\mu, \sigma^2)$ sample with both μ and σ^2 unknown. We want to test H_0: $\sigma = \sigma_0$ vs. H_1: $\sigma \neq \sigma_0$.

Show that the size α likelihood ratio test accepts if, and only if

$$c_1 \leq \frac{1}{\sigma_0^2} \sum_{i=1}^{n} (X_i - \bar{X})^2 \leq c_2$$

where c_1 and c_2 satisfy

1. $F(c_2) - F(c_1) = 1 - \alpha$, where F is the c.d.f. of the χ^2_{n-1} distribution.

2. $c_1 - c_2 = n \log c_1 / c_2$.