STAT 215 – Midterm 2 NAME:

1. We have data for trees growing in the rainforest in Peruvian Amazonia. Let x_i be the average growth rate over the first five years. Let T_i denote the total life time of a tree. A traditional model for tree survival times assumes an exponential regression of survival times T_i on x_i:

$$T_i \sim Exp(\mu), \quad \mu = \gamma_0 e^{\gamma_1 x_i}, \ \gamma_0 > 0,$$

i.e., $p(T_i|\gamma_0, \gamma_1, x_i) = \mu \exp(-\mu T_i) = \gamma_0 e^{\gamma_1 x_i} \exp(-\gamma_0 e^{\gamma_1 x_i} T_i)$.
You might also need the exponential c.d.f.: $Pr(T_i \le S) = 1 - \exp(-\mu S)$.

Data: For the first n trees we observe the actual survival times T_i, for the last m trees we only observe censored survival times, i.e., we only know that $T_i \ge S_i$ for some S_i.
Thus the data are $T_i, i = 1, \ldots, n$, and $S_i, i = n+1, \ldots, n+m$. For all trees, $i = 1, \ldots, n+m$, we know x_i. We will denote the data with $x = (x_1, \ldots, x_{n+m})$, $T = (T_1, \ldots, T_n)$ and $S = (S_{n+1}, \ldots, S_{n+m})$.

Prior: We consider two alternative models:

M1: $\gamma_1 = 1.0$ and $p(\gamma_0) = Ga(\alpha, \beta)$. *M2:* $\gamma_1 = \frac{1}{2}$ and $p(\gamma_0) = Ga(\alpha, \beta)$.

1a. [5pts] Give the joint likelihood $p(T, S|x, \gamma_0, \gamma_1)$.

Let $R_1 = \sum_{i=1}^n x_i$, $U_1(\gamma_1) = \sum_{i=1}^n e^{\gamma_1 x_i} T_i$ and $U_2(\gamma_1) = \sum_{i=n+1}^{n+m} e^{\gamma_1 x_i} S_i$.

$$p(T, S|x, \gamma_0, \gamma_1) = \prod_{i=1}^{n} \gamma_0 e^{\gamma_1 x_i} e^{-\gamma_0 e^{\gamma_1 x_i} T_i} \prod_{i=n+1}^{n+m} e^{-\gamma_0 e^{\gamma_1 x_i} S_i} = \gamma_0^n e^{\gamma_1 R_1} e^{-\gamma_0 [U_1(\gamma_1) + U_2(\gamma_1)]}$$

1b. Find the marginal distribution of the data (T, S) under model M1, i.e., $p(T, S|x, M1) = p(T, S|x, \gamma_1 = 1)$. [5pts]

$$\begin{aligned} p(T, S|x, \gamma_1 = 1) &= \int p(T, S|x, \gamma_0, \gamma_1) p(\gamma_0) d\gamma_0 = \frac{\beta^\alpha}{\Gamma(\alpha)} \int \gamma_0^n e^{R_1} e^{-\gamma_0 [U_1(1) + U_2(1)]} \gamma_0^{(\alpha-1)} e^{-\beta \gamma_0} d\gamma_0 \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \Gamma(\alpha + n) e^{R_1} . [\beta + U_1(1) + U_2(1)]^{-(n+\alpha-1)} \end{aligned}$$

1c. Find the marginal distribution of the data (T, S) under model M2, i.e., $p(T, S|x, M2) = p(T, S|x, \gamma_1 = \frac{1}{2})$. [5pts]

$$p(T, S|x, \gamma_1 = \frac{1}{2}) = \frac{\beta^\alpha}{\Gamma(\alpha)} \Gamma(\alpha + n) e^{\frac{1}{2} R_1} . \left[\beta + U_1(\frac{1}{2}) + U_2(\frac{1}{2})\right]^{-(n+\alpha-1)}$$

1d. [5pts] Find the Bayes factor B_{12} for model M2 against M1.

$$B_{12} = \left[\frac{\beta + U_1(1) + U_2(1)}{\beta + U_1(\frac{1}{2}) + U_2(\frac{1}{2})}\right]^{(n+\alpha-1)} e^{-\frac{1}{2} R_1}.$$

1e. [5pts] Consider a third model *M3:* $Pr(\gamma_1 = \frac{j}{4}) = \frac{1}{4}$ for $j = 1, \ldots, 4$, i.e., a discrete prior on γ_1 with equal point masses at $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$ and 1.0. And, again, the same Gamma prior on γ_0, $p(\gamma_0) = Ga(\alpha, \beta)$, with γ_0 and γ_1 *a priori* independent.

Find the marginal distribution of the data (S, T) under model M3, i.e., $p(T, S|x, M3)$.

$$p(T, S|x, M3) = \frac{\beta^\alpha}{\Gamma(\alpha)} \Gamma(\alpha + n) \frac{1}{4} \sum_{j=1}^{4} e^{\frac{j}{4} R_1} \left[\beta + U_1(\frac{j}{4}) + U_2(\frac{j}{4})\right]^{-(n+\alpha-1)}$$

2. Consider the following hierarchical model:

$$\begin{aligned} y_i &= x_i'\beta + z_i'\gamma + \epsilon_i, \\ & \quad \epsilon_i \overset{iid}{\sim} N(0,\sigma^2), \quad i = 1,\ldots,n \\ \beta &\sim N(\mu,\Sigma), \quad \gamma \sim N(m_\gamma, S_\gamma), \\ \mu &\sim N(m_\beta, S_\beta), \end{aligned}$$

where x_i, and β are $(p \times 1)$ vectors, z_i and γ are $(q \times 1)$ vectors. Assume $m_\gamma, S_\gamma, m_\beta, S_\beta$ and σ^2 are known, and β and γ are *a priori* independent. Also, x_i and z_i are known covariate vectors.

2a. [10pts] Denote with X the $(n \times p)$ matrix with x_i' in the i-th row, and with Z the $(n \times q)$ matrix with z_i in the i-th row. Let $y = (y_1, \ldots, y_n)'$. Prove

$$p(\beta|\mu,\gamma,y) = N(b,B)$$

with $B^{-1} = \Sigma^{-1} + 1/\sigma^2(X'X)$ and $b = B\left[\Sigma^{-1}\mu + 1/\sigma^2 X'(y - Z\gamma)\right]$.

Rewrite the model as: $y = X\beta + Z\gamma + \epsilon$, $\epsilon \sim N(0,\sigma^2 I)$ and multiply on both sides with $(X'X)^{-1}X'$. Let $\tilde{y} = (X'X)^{-1}X'(y - Z\gamma)$.

$$\begin{aligned} \tilde{y} &\sim N(\beta, (X'X)^{-1}\sigma^2) \\ \beta &\sim N(\mu,\Sigma). \end{aligned}$$

By (3.18) (textbook): $p(\beta|\mu,\gamma,y) = N(b,B)$ with $B^{-1} = \Sigma^{-1} + \sigma^{-2}(X'X)$ and

$$b = B\left[\Sigma^{-1}\mu + \sigma^{-2} \underbrace{(X'X)\tilde{y}}_{=X'(y-Z\gamma)}\right].$$

2b. [5pts] Find $p(y|\gamma,\mu)$ and $p(\gamma|\mu,y)$.

$p(y|\gamma,\mu) = N(X\mu + Z\gamma, \sigma^2 I + X\Sigma X')$ (see, for example, solution of homework 10).
Let $H = (Z'Z)^{-1}Z'$ and $\tilde{y} = H(y - X\mu)$.

$$\begin{aligned} \tilde{y} &\sim N(\gamma, \underbrace{H(\sigma^2 I + X\Sigma X')H'}_{A}), \\ \gamma &\sim N(m_\gamma, S_\gamma). \end{aligned}$$

Then, by (3.18): $p(\gamma|\mu,y) \sim N(c,C)$ with $C^{-1} = S_\gamma^{-1} + A^{-1}$ and $c = C(S_\gamma^{-1}m_\gamma + A^{-1}\tilde{y})$.

Note: By a similar argument we could find $p(y|\mu)$ and $p(\mu|y)$. Using $p(\mu|y), p(\gamma|\mu,y)$ and $p(\beta|\mu,\gamma,y)$ we could then generate i.i.d. samples from the posterior distribution.

Alternatively one could use iterative simulation from complete conditionals using $p(\beta|\mu,\gamma,y)$, $p(\gamma|\beta,\mu,y)$ and $p(\mu|\beta,\gamma,y)$. The next two questions ask for the relevant conditional distributions.

2c. [5pts] Find $p(\gamma|\mu,\beta,y)$.

Same as **2a**, with β and γ switching roles: $p(\gamma|\mu,\beta,y) = N(d,D)$ with $D^{-1} = S_\gamma^{-1} + 1/\sigma^2(Z'Z)$, and $d = D\left[S_\gamma^{-1}m_\gamma + 1/\sigma^2 Z'(y - X\beta)\right]$.

2d. [5pts] Find $p(\mu|\beta,\gamma,y)$.

The relevant parts of the model are:

$$\beta = N(\mu,\Sigma), \quad \mu \sim N(m_\beta, S_\beta)$$

(by 3.18) $\Rightarrow p(\mu|\beta,\gamma,y) = N(e,E)$ with $E^{-1} = \Sigma^{-1} + S_\beta^{-1}$ and $e = E(S_\beta^{-1}m_\beta + \Sigma^{-1}\beta)$.