Solutions for Exercises 4.1 and 4.3

4.1: *Obtain an example of a model and loss function for which there is a* θ_0 *such that* $\delta(X) = \theta_0$ *is not admissibile.* Many possibilities exist, the most straightforward being those where the distributions in the model $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ do not have common support. For example, suppose there are θ_0 and θ_1 in Θ, where the supports S_{θ_0} and S_{θ_1} of P_{θ_0} and P_{θ_1} have no overlap, so that $P_{\theta_1}(S_{\theta_0}) = 0 = P_{\theta_0}(S_{\theta_1})$. Then the estimator $\delta'(x) = \theta_0 1(x \in S_{\theta_0}) + \theta_1 1(x \in S_{\theta_1})$ dominates the constant estimator $\delta(x) = \theta_0$.

4.3:

(a) The four non-randomized estimators are

$$\begin{aligned}
\delta_{ll}(x) &= \theta_l \\
\delta_{lh}(x) &= (1-x)\theta_l + x\theta_h \\
\delta_{hl}(x) &= x\theta_h + (1-x)\theta_l \\
\delta_{hh}(x) &= \theta_h
\end{aligned}$$

(b) The randomized estimators can be expressed as

$$\delta_r(x, U) = \begin{cases} \delta_{ll}(x) & \text{if } 0 < U < a \\ \delta_{lh}(x) & \text{if } a < U < b \\ \delta_{hl}(x) & \text{if } b < U < c \\ \delta_{hh}(x) & \text{if } c < U < 1 \end{cases}$$

where $U \sim$ uniform(0,1). The risk set has a diamond shape, with the risks of the non-randomized estimators as vertices. These risks are $(0,1)$ for δ_{ll}, $(1-\theta_l, \theta_h)$ for δ_{hl}, $(1,0)$ for δ_{hh} and $(\theta_l, 1-\theta_h)$ for δ_{lh}.

(c) The admissibile estimators are on the "southwest" boundary of the risk set. This inclues δ_{ll}, δ_{lh} and δ_{hh}, and randomized estimators that are combinations of δ_{lh} and one of δ_{ll} and δ_{hh}.

(d) (a) If $\pi(\theta_l) > \theta_h/(\theta_l + \theta_h)$, then δ_{ll} is Bayes.

(b) If $\pi(\theta_l) = \theta_h/(\theta_l + \theta_h)$, then randomized combinations of δ_{ll} and δ_{lh} are Bayes.

(c) If $\theta_h/(\theta_l + \theta_h) < \pi(\theta_l) < (1-\theta_h)/(2 - (\theta_l + \theta_h))$, then δ_{lh} is Bayes.

(d) If $\pi(\theta_l) = (1-\theta_h)/(2-(\theta_l+\theta_h))$, then randomized combinations of δ_{hh} and δ_{lh} are Bayes.

(e) If $\pi(\theta_l) < (1-\theta_h)/(2-(\theta_l+\theta_h))$, then δ_{hh} is Bayes.