

Estimation of group effects

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Duke STA 610

Bias, variance and MSE

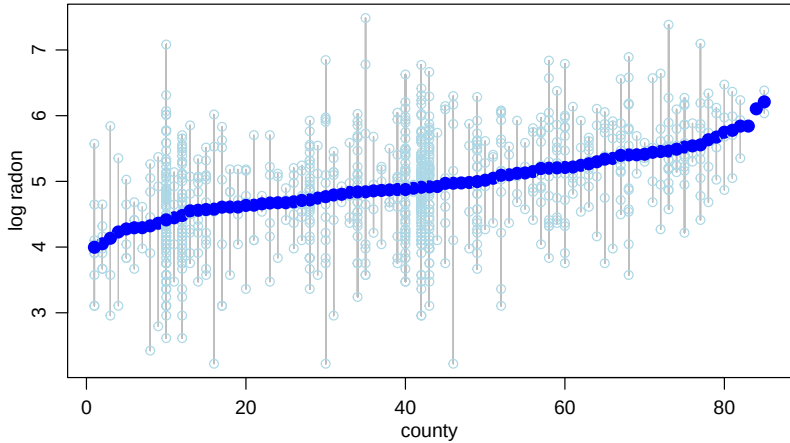
Fixed groups perspective

Random groups perspective

Bayesian perspective

Bayesian perspective

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Different amounts of information

```
y[g=="LACQUIPARLE"]  
## [1] 6.036210 6.383751  
  
y[g=="WASHINGTON"]  
  
## [1] 5.933906 5.653191 4.412045 5.484196 6.112774 5.139915 5.437089 5.484196  
## [9] 4.648416 4.269652 3.834061 4.497065 3.668259 3.834061 4.104487 3.473607  
## [17] 4.162503 5.161298 4.162503 4.810531 3.473607 5.893950 5.280842 5.751848  
## [25] 4.269652 5.499419 4.950219 5.387661 5.202746 4.537062 5.981707 4.497065  
## [33] 4.366735 5.161298 4.923785 6.206521 4.682979 5.072896 4.950219 4.217459  
## [41] 4.043070 4.217459 3.908367 5.499419 6.626603 5.404409
```

Linear shrinkage estimator: $\hat{\theta}_j = (1 - w_j)\bar{y}_j + w_j c$

- What should c be?
- What should w_j depend on?

Mean squared error

- Let θ be the subpopulation mean of a generic group;
- let $\hat{\theta}$ be an estimator of θ (a function of the data).

The *mean squared error* (MSE) of $\hat{\theta}$ is

$$MSE[\hat{\theta}|\theta] = E[(\hat{\theta} - \theta)^2|\theta]$$

Bias-variance decomposition: Let $m(\theta) = E[\hat{\theta}|\theta]$.

$$\begin{aligned} MSE[\hat{\theta}|\theta] &= E[(\hat{\theta} - m + m - \theta)^2|\theta] \\ &= E[(\hat{\theta} - m)^2|\theta] + 2E[(\hat{\theta} - m)(m - \theta)|\theta] + E[(m - \theta)^2|\theta] \\ &= E[(\hat{\theta} - m)^2|\theta] + (m - \theta)^2 \\ &= \text{Var}[\hat{\theta}|\theta] + \text{Bias}^2[\hat{\theta}|\theta] \end{aligned}$$

Bias-variance tradeoff

In general,

$$MSE[\hat{\theta}|\theta] = \text{Var}[\hat{\theta}|\theta] + \text{Bias}(\hat{\theta}|\theta)^2$$

How well an estimator $\hat{\theta}$ does at estimating θ depends on *variance* and *bias*.

In general,

- estimators with low bias have high variance ($\hat{\theta} = \bar{y}$ but small n);
- estimators with low variance have high bias ($\hat{\theta} = 5$).

Minimizing MSE requires balancing bias and variance.

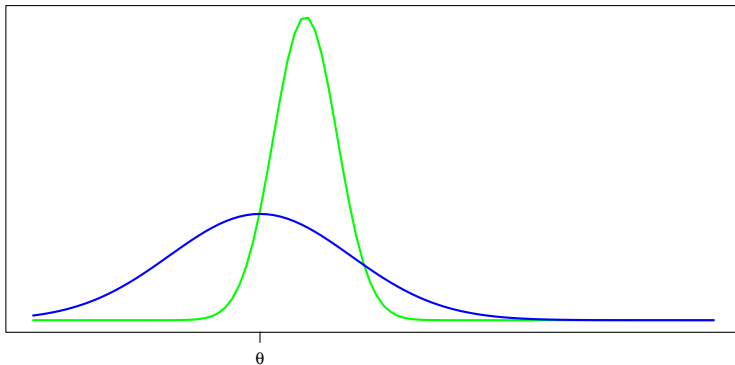
Bias, variance and MSE
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Fixed groups perspective
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Random groups perspective
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Bayesian perspective
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Bias-variance tradeoff



Sample mean bias and variance

Let $y_1, \dots, y_n$ be sample from a population with mean θ , variance σ^2 .

Sample mean estimator: Let $\hat{\theta} = \bar{y}$

$$E[\bar{y}|\theta] = \theta$$

$$\text{Bias}[\bar{y}|\theta] = 0$$

$$\text{Var}[\bar{y}|\theta] = \sigma^2/n$$

$$\text{MSE}[\bar{y}|\theta] = \text{Var}[\bar{y}|\theta] = \sigma^2/n$$

Linear shrinkage bias and variance

Linear shrinkage estimator: $\hat{\theta} = (1 - w)\bar{y} + wc$ for some $w \in [0, 1]$.

- w is the amount of shrinkage;
- c is the shrinkage point.

$$E[\hat{\theta}|\theta] = (1 - w)\theta + wc = \theta + w(c - \theta)$$

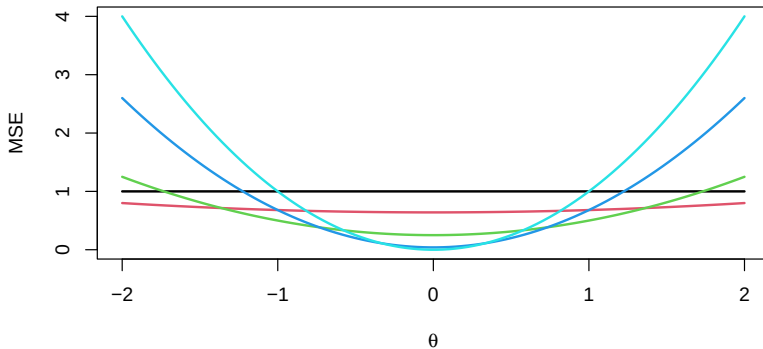
$$\text{Bias}[\hat{\theta}|\theta]^2 = w^2(c - \theta)^2 \geq 0$$

$$\text{Var}[\hat{\theta}|\theta] = (1 - w)^2\sigma^2/n \leq \sigma^2/n$$

$$MSE[\hat{\theta}|\theta] = (1 - w)^2\sigma^2/n + w^2(c - \theta)^2$$

Mean squared error function

$$\sigma^2/n = 1 \quad c = 0$$



Composite MSE

Consider a LSE for $\boldsymbol{\theta} = (\theta_1, \dots, \theta_m)$ where $\hat{\theta}_j = (1 - w)\bar{y}_j + wc$

$$\begin{aligned}MSE[\hat{\boldsymbol{\theta}}|\boldsymbol{\theta}] &= E[||\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}||^2|\boldsymbol{\theta}] \\&= \sum_j E[(\hat{\theta}_j - \theta_j)^2|\boldsymbol{\theta}] \\&= \frac{\sigma^2}{n}m(1 - w)^2 + w^2 \sum_j (c - \theta_j)^2\end{aligned}$$

What should the values of w and c be?

Oracle estimator

Using calculus you can show that MSE is optimized by

- $c = \mu = \sum_j \theta_j / m$;
- $w = \frac{1/\tau^2}{n/\sigma^2 + 1/\tau^2}$, where
- $\tau^2 = \sum_j (\theta_j - \mu)^2 / m$.

This gives the *oracle estimator*

$$\hat{\theta}_j = \frac{n/\sigma^2}{n/\sigma^2 + 1/\tau^2} \bar{y}_j + \frac{1/\tau^2}{n/\sigma^2 + 1/\tau^2} \mu.$$

This can also be written

$$\hat{\theta}_j = \frac{\tau^2}{\sigma^2/n + \tau^2} \bar{y}_j + \frac{\sigma^2/n}{\sigma^2/n + \tau^2} \mu.$$

Composite risk comparison

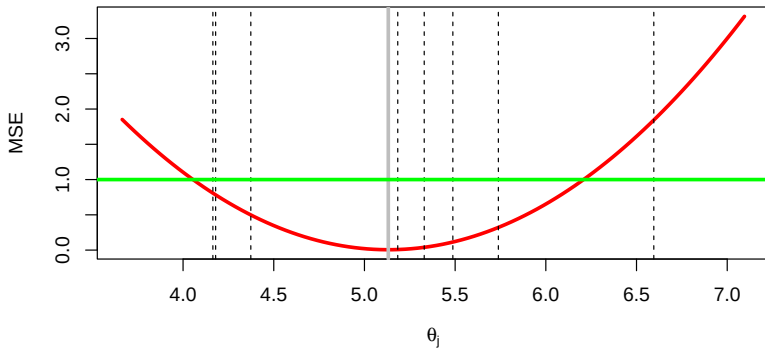
- $\bar{\mathbf{y}} = (\bar{y}_1, \dots, \bar{y}_m)$, the vector of sample means;
- $\hat{\boldsymbol{\theta}} = (\hat{\theta}_1, \dots, \hat{\theta}_m)$, the vector of oracle estimates.

$$MSE[\bar{\mathbf{y}}|\boldsymbol{\theta}] = m \frac{\sigma^2}{n}$$
$$MSE[\hat{\boldsymbol{\theta}}|\boldsymbol{\theta}] = m \frac{\sigma^2}{n} \times \left(\frac{\tau^2}{\sigma^2/n + \tau^2} \right) < MSE[\bar{\mathbf{y}}|\boldsymbol{\theta}].$$

The oracle estimator is better than $\bar{\mathbf{y}}$ *in terms of composite risk* .

Group-level risk of oracle estimator

$$MSE[\hat{\theta}_j|\theta] = (1-w)^2\sigma^2/n + w^2(\theta_j - \mu)^2.$$



Summary

Composite risk

- $\bar{y}$ is an unbiased estimator of θ ;
- $\hat{\theta}$ is a biased estimator of θ , but has lower variance than $\bar{y}$.
- $MSE[\hat{\theta}|\theta] \leq MSE[\bar{y}|\theta]$.

Group-level risk

- $\bar{y}_j$ is an unbiased estimator of θ_j for each $j = 1, \dots, m$.
- $\hat{\theta}_j$ is a biased estimator of θ_j , but has lower variance than $\bar{y}_j$.
- $MSE[\hat{\theta}_j|\theta] \gtrless MSE[\bar{y}_j|\theta]$ and you don't know which!

Practical considerations

Typically,

- μ, τ^2, σ^2 are unknown;
- sample sizes may vary across groups.

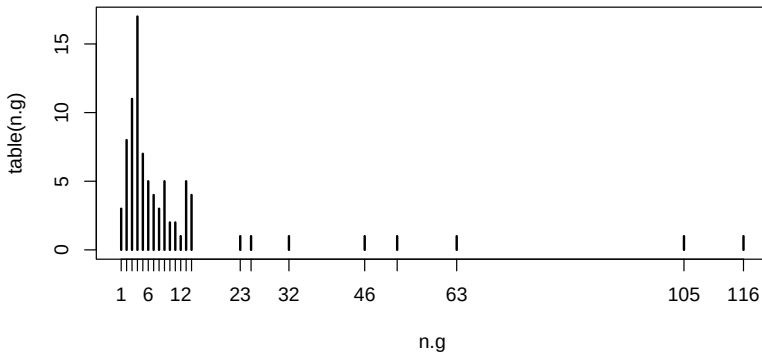
In practice, people use the following *adaptive shrinkage estimator*:

$$\hat{\theta}_j = \frac{n_j/\hat{\sigma}^2}{n_j/\hat{\sigma}^2 + 1/\hat{\tau}^2} \bar{y}_j + \frac{1/\hat{\tau}^2}{n_j/\hat{\sigma}^2 + 1/\hat{\tau}^2} \hat{\mu}.$$

- $\hat{\mu}, \hat{\tau}^2, \hat{\sigma}^2$ are obtained from the data (e.g. ANOVA or lme4).
- If $n_j = n$, can obtain $\hat{\mu}, \hat{\tau}^2, \hat{\sigma}^2$ so that $\hat{\theta}$ is guaranteed better than $\bar{y}$ (Stein).
- Otherwise, for large m , $\hat{\theta}$ will be approximately optimal linear estimator (under composite risk).

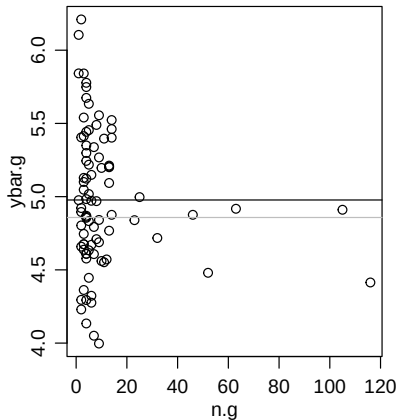
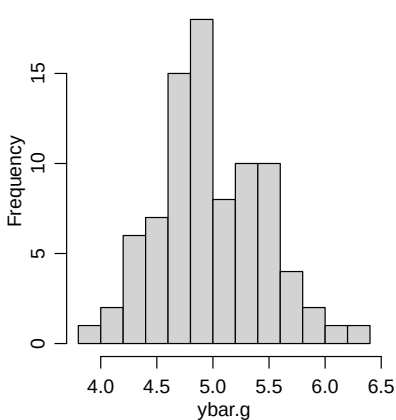
Radon example

```
n.g<-c(table(g) )  
plot(table(n.g))
```



Radon example

```
# county specific radon means  
ybar.g <- c(tapply(y, g, "mean"))
```



MLEs

```
library(lme4)
fit.lme<-lmer(y~1+(1|g),REML=FALSE)
summary(fit.lme)

## Linear mixed model fit by maximum likelihood ['lmerMod']
## Formula: y ~ 1 + (1 | g)
##
##           AIC          BIC    logLik deviance df.resid
##    2164.1    2178.5   -1079.0    2158.1         916
##
## Scaled residuals:
##      Min       1Q   Median       3Q      Max
## -3.6165 -0.6141  0.0292  0.6526  3.4932
##
## Random effects:
##   Groups      Name            Variance Std.Dev.
##   g          (Intercept)  0.08804   0.2967
##   Residual                0.57154   0.7560
## Number of obs: 919, groups:  g, 85
##
## Fixed effects:
##              Estimate Std. Error t value
## (Intercept)  4.94656    0.04664   106.1
```

Parameter estimates

```
VarCorr(fit.lme)

## Groups      Name      Std.Dev.
## g          (Intercept) 0.29672
## Residual                    0.75600

t2.mle<-as.numeric(VarCorr(fit.lme)$g)

t2.mle

## [1] 0.08804027

sigma(fit.lme)

## [1] 0.7559996

s2.mle<-sigma(fit.lme)^2

s2.mle

## [1] 0.5715354

fixef(fit.lme)

## (Intercept)
##      4.946557

mu.mle<-fixef(fit.lme)
```

Adaptive shrinkage estimates

Replace μ, σ^2, τ^2 with estimates:

$$\hat{\theta}_j = w_j \bar{y}_j + (1 - w_j) \hat{\mu}, \text{ where } w_j = \frac{n_j / \hat{\sigma}^2}{n_j / \hat{\sigma}^2 + 1 / \hat{\tau}^2}.$$

```
w.shrink<- (n.g/s2.mle) /(n.g/s2.mle + 1/t2.mle)

mu.shrink<-w.shrink*ybar.g + (1-w.shrink)*mu.mle

mu.mle

## (Intercept)
##      4.946557

cbind(ybar.g, n.g, mu.shrink)[1:8,]

##           ybar.g n.g mu.shrink
## AITKIN      4.293832   4  4.697704
## ANOKA       4.479973  52  4.531757
## BECKER      4.675008   3  4.860730
## BELTRAMI    4.793035   7  4.866904
## BENTON      4.869503   4  4.917180
## BIGSTONE    5.128199   3  5.003968
## BLUEEARTH   5.522876  14  5.340299
## BROWN       5.244160   4  5.060018
```

Shrinkage

```
topten<-order(ybar.g,decreasing=TRUE)[1:10]  
cbind(ybar.g, n.g, mu.shrink)[topten,]
```

##	ybar.g	n.g	mu.shrink
## LACQUIPARLE	6.209980	2	5.244122
## MURRAY	6.104550	1	5.101126
## WILKIN	5.841654	1	5.066035
## WATONWAN	5.841041	3	5.229271
## NICOLLET	5.777273	4	5.263269
## LINCOLN	5.748294	4	5.252221
## KANDIYOHI	5.674289	4	5.224006
## JACKSON	5.633758	5	5.245555
## FREEBORN	5.555495	9	5.300322
## NOBLES	5.540083	3	5.134149

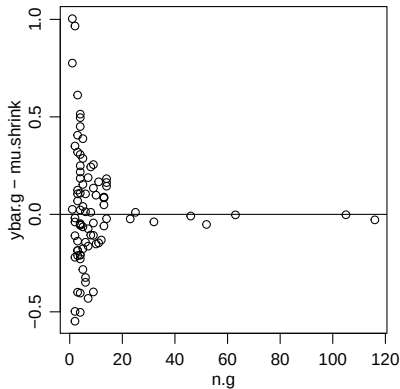
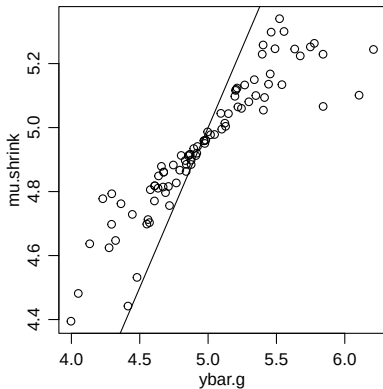
Bias, variance and MSE
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Fixed groups perspective
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Random groups perspective
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Bayesian perspective
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Shrinkage



Shrinkage estimates from lme4

```
mu.shrink[1:10]

##      AITKIN      ANOKA      BECKER  BELTRAMI      BENTON  BIGSTONE BLUEEARTH      BROWN
## 4.697704 4.531757 4.860730 4.866904 4.917180 5.003968 5.340299 5.060018
##   CARLTON   CARVER
## 4.712463 4.958725

a.shrink<-ranef(fit.lme)[[1]][,1]

mu.mle+a.shrink[1:10]

## [1] 4.697704 4.531757 4.860730 4.866904 4.917180 5.003968 5.340299 5.060018
## [9] 4.712463 4.958725
```

In `lme4`, `ranef(fit.lme)[[k]][,1]` refers to the

- 1th random effect for the
- kth grouping variable.

Hierarchical normal model

$$y_{i,j} = \mu + a_j + \epsilon_{i,j}$$

$$\{\epsilon_{1,1}, \dots, \epsilon_{n,1}\}, \dots, \{\epsilon_{1,m}, \dots, \epsilon_{n,m}\} \sim \text{i.i.d. normal}(0, \sigma^2)$$

$$a_1, \dots, a_m \sim \text{i.i.d. normal}(0, \tau^2)$$

Equivalently,

$$y_{i,j} = \theta_j + \epsilon_{i,j}$$

$$\{\epsilon_{1,1}, \dots, \epsilon_{n,1}\}, \dots, \{\epsilon_{1,m}, \dots, \epsilon_{n,m}\} \sim \text{i.i.d. normal}(0, \sigma^2)$$

$$\theta_1, \dots, \theta_m \sim \text{i.i.d. normal}(\mu, \tau^2)$$

In this model, we think of

- the groups as being randomly selected from a larger set of possible groups,
- so the means are randomly selected from a set of possible means.
- This interpretation is *not appropriate* for the radon data!

Unbiased predictors

Suppose you will sample a random group with subgroup mean θ , so

$$\theta \sim N(\mu, \tau^2)$$

$$\bar{y}|\theta \sim N(\theta, \sigma^2/n).$$

How should you plan on estimating θ ? Consider estimators $\tilde{\theta}$ that are unbiased “on average:”

$$E[\tilde{\theta} - \theta] = \int_{\theta} \left(\int_y (\hat{\theta} - \theta) p(y|\theta) dy \right) p(\theta|\mu, \tau^2) d\theta$$

- Such estimators are sometimes called “unbiased predictors”;
- They might not be unbiased for most values of θ !

Unbiased predictors

$$\bar{y} = \sum y_i / n$$
$$\hat{\theta} = \frac{\tau^2}{\sigma^2/n + \tau^2} \bar{y} + \frac{\sigma^2/n}{\sigma^2/n + \tau^2} \mu.$$

Exercises:

1. Show that $\bar{y}$ is an unbiased predictor;
2. Show that $\hat{\theta}$ is an unbiased predictor.
3. Identify some other unbiased predictors.

Best unbiased prediction

Result 1: (Best unbiased predictor). Let $\tilde{\theta}$ be any unbiased predictor, meaning $E[\tilde{\theta} - \theta] = 0$ where the expectation is averaging over *both* y and θ . Then

$$E[(\hat{\theta} - \theta)^2] \leq E[(\tilde{\theta} - \theta)^2]$$

where the expectation is over *both* y and θ .

Best linear unbiased predictor

A similar result holds even if the data are not normal. Suppose

- $E[\bar{y}|\theta] = \theta$, $\text{Var}[\bar{y}|\theta] = \sigma^2/n$.
- $E[\theta] = \mu$, $\text{Var}[\theta] = \tau^2$.

Result 2: (Best linear unbiased predictor). Let $\tilde{\theta}$ be any *linear* unbiased predictor, meaning

- $\tilde{\theta} = a\bar{y} + b$ for some fixed a and b ;
- $E[\tilde{\theta} - \theta] = 0$ where the expectation is averaging over *both* y and θ .

Then

$$E[(\hat{\theta} - \theta)^2] \leq E[(\tilde{\theta} - \theta)^2]$$

where the expectation is over *both* y and θ .

BLUPs

The $\hat{\theta}_j$'s are often called the *best unbiased linear predictors (BLUPs)*.

This is confusing, as we have discussed how these estimators are biased:

$$\begin{aligned} E[\hat{\theta}_j | \theta_j] &= E[w\bar{y}_j + (1 - w)\mu | \theta_j] \\ &= w\theta_j + (1 - w)\mu \neq \theta_j \end{aligned}$$

$\hat{\theta}_j$ is *conditionally* biased.

The “U” in BLUP refers to bias only in an unconditional sense:

$$\begin{aligned} E[\hat{\theta}_j] &= E[E[\hat{\theta}_j | \theta_j]] \\ &= E[w\theta_j + (1 - w)\mu] \\ &= w\mu + (1 - w)\mu = \mu. \end{aligned}$$

Since $E[\hat{\theta}_j] = E[\theta_j] = \mu$ *unconditionally*, people might say $\hat{\theta}_j$ is “unbiased.”

Understanding conditional and unconditional expectation

school	A	B	C	D	E	F	G	H	I	J
mean	θ_A	θ_B	θ_C	θ_D	θ_E	θ_F	θ_G	θ_H	θ_I	θ_J

Let $\mu = (\theta_A + \cdots \theta_J)/10$.

Study design:

- sample m schools at random from the population of schools.
- sample n students at random from each of the m schools.

What is the expectation of θ_1 , $\bar{y}_1$, $\hat{\theta}_1$?

Expectation of θ_1 : Since each school A through J has equal probability of being selected as unit 1:

$$\begin{aligned} E[\theta_1] &= \theta_A \times \Pr(\text{unit } 1 = A) + \cdots + \theta_J \times \Pr(\text{unit } 1 = J) \\ &= \theta_A \frac{1}{10} + \cdots + \theta_J \frac{1}{10} = \mu \end{aligned}$$

Understanding conditional expectation

$$E[\bar{y}_1 - \theta_1 | \text{unit 1} = D] = E[\bar{y}_D - \theta_D] = \theta_D - \theta_D = 0$$

$$\begin{aligned} E[\hat{\theta}_1 - \theta_1 | \text{unit 1} = D] &= E[(1 - w)\bar{y}_D + w\mu - \theta_D] \\ &= (1 - w)\theta_D + w\mu - \theta_D = w(\mu - \theta_D) \neq 0 \end{aligned}$$

Conditionally on *unit 1=D*,

- $\bar{y}_1 = \bar{y}_D$ is unbiased for θ_D ,
- $\hat{\theta}_1 = \hat{\theta}_D$ is biased for θ_D .

In English, if your first sampled school is school D, then

- $\bar{y}_1 = \bar{y}_D$ and $\bar{y}_D$ is unbiased for θ_D
- $\hat{\theta}_1 = \hat{\theta}_D$ and $\hat{\theta}_D$ is biased for θ_D .

Understanding unconditional expectation

Before you sample the schools, unit 1 is equally likely to be school A, B, ..., J.

$$\begin{aligned} E[\hat{\theta}_1 - \theta_1] &= E[\hat{\theta}_A - \theta_A] \Pr(\text{unit 1}=A) + \cdots + E[\hat{\theta}_J - \theta_J] \Pr(\text{unit 1}=J) \\ &= w(\mu - \theta_A) \times \frac{1}{10} + \cdots + w(\mu - \theta_J) \times \frac{1}{10} \\ &= w\mu - w(\theta_A + \cdots + \theta_J) \frac{1}{10} \\ &= w\mu - w\mu = 0. \end{aligned}$$

This unconditional expectation, and the “U” in BLUP, refers to averaging across the possibilities for the samples:

- $\hat{\theta}_j$ will be a biased estimator of the mean of whatever unit is picked j th.
- on average across studies, $\hat{\theta}_1, \dots, \hat{\theta}_m$ will be unbiased.

Practical considerations

As before,

- μ, τ^2, σ^2 are unknown;
- sample sizes may vary across groups.

In practice, people use the following *Empirical BLUP*:

$$\hat{\theta}_j = \frac{n_j/\hat{\sigma}^2}{n_j/\hat{\sigma}^2 + 1/\hat{\tau}^2} \bar{y}_j + \frac{1/\hat{\tau}^2}{n_j/\hat{\sigma}^2 + 1/\hat{\tau}^2} \bar{\theta},$$

where $\hat{\mu}, \hat{\tau}^2, \hat{\sigma}^2$ are estimated from the data (ANOVA or lme4)

This is the same estimator as the adaptive shrinkage estimator.

- variability in “random” θ_j ’s $\approx$ heterogeneity in “fixed” θ_j ’s.
- integration over θ_j ’s to get MSE $\approx$ summing over θ_j ’s to get composite MSE.

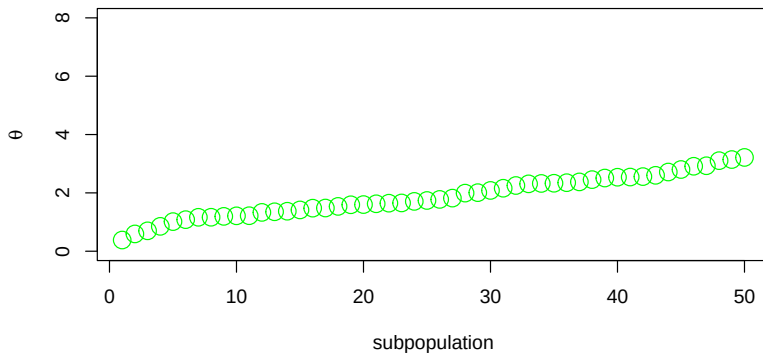
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Fixed groups perspective
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Random groups perspective
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Bayesian perspective
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Simulation Example



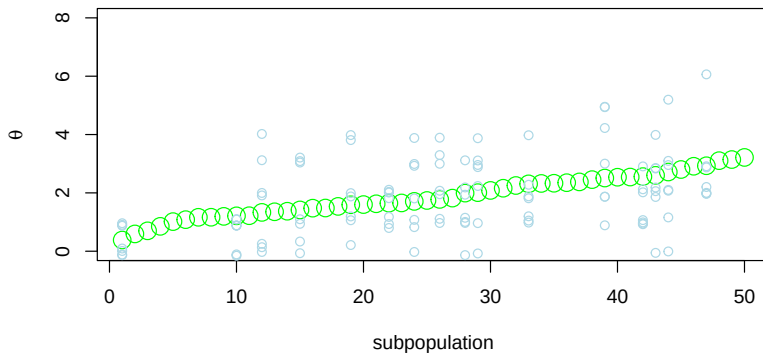
Bias, variance and MSE
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Fixed groups perspective
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Random groups perspective
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Bayesian perspective
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Simulation Example



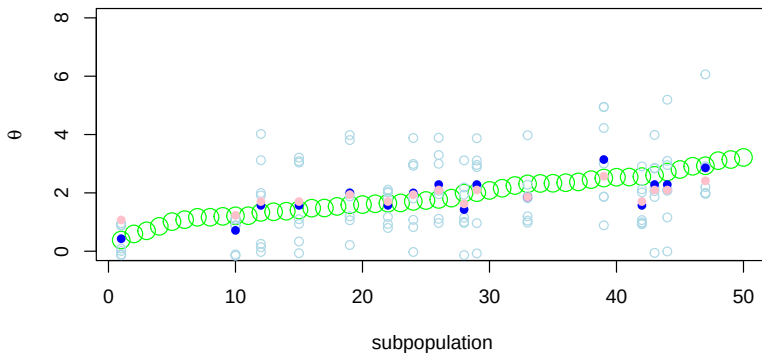
Bias, variance and MSE
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Fixed groups perspective
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Random groups perspective
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Bayesian perspective
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Simulation Example



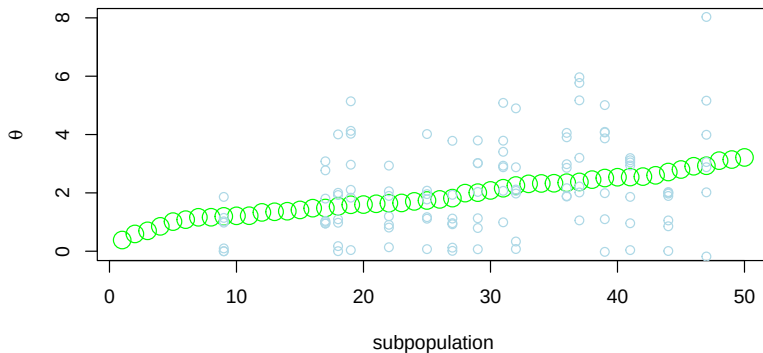
Bias, variance and MSE
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Fixed groups perspective
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Random groups perspective
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Bayesian perspective
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Simulation Example



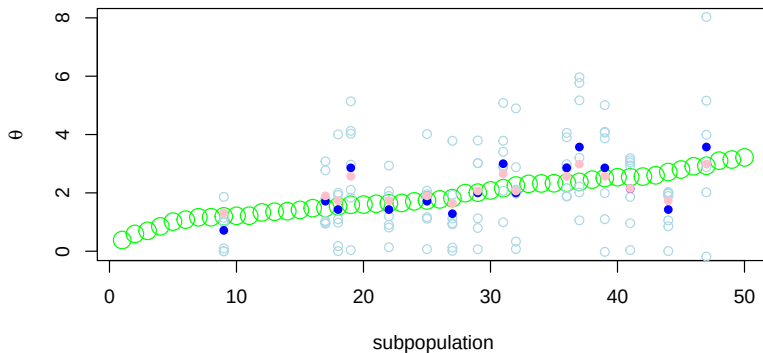
Bias, variance and MSE
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Fixed groups perspective
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Random groups perspective
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Bayesian perspective
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Simulation Example



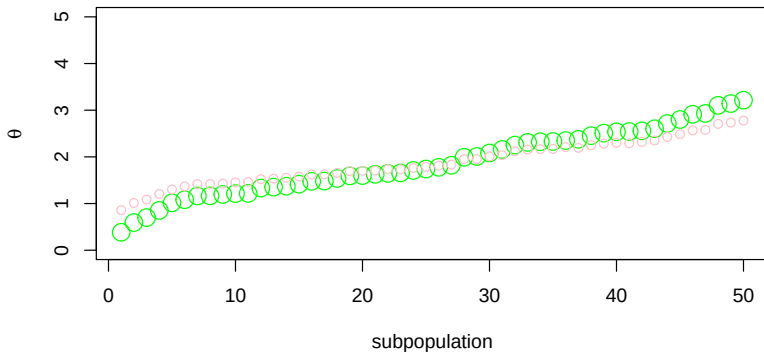
Bias, variance and MSE
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Fixed groups perspective
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Random groups perspective
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Bayesian perspective
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Expectation of estimators



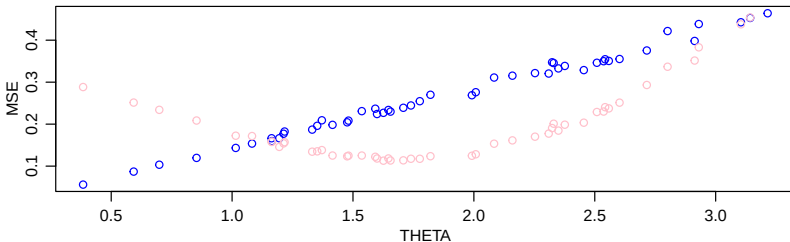
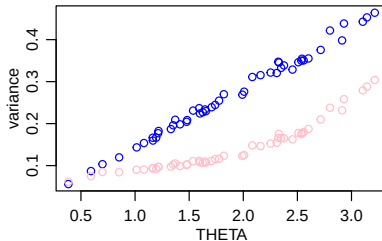
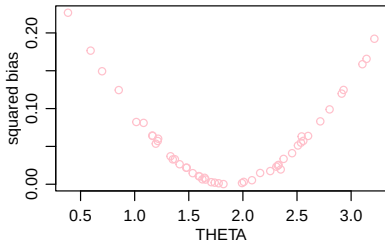
Bias, variance and MSE
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Fixed groups perspective
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Random groups perspective
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Bayesian perspective
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Bias squared, Variance and MSE



Review of Bayesian inference

- Prior density: $p(\gamma)$
- Sampling density: $p(y_1, \dots, y_n | \gamma)$

The prior density describes where you think γ is, before having seen the data.

The sampling density describes where you think the data will be, for each possible value of γ .

Bayes rule:

$$\begin{aligned} p(\gamma | y_1, \dots, y_n) &= \frac{p(\gamma)p(y_1, \dots, y_n | \gamma)}{\int p(\gamma')p(y_1, \dots, y_n | \gamma') d\gamma'} \\ &\propto p(\gamma)p(y_1, \dots, y_n | \gamma) \end{aligned}$$

The *posterior density* $p(\gamma | y_1, \dots, y_n)$ describes where you think the γ is, after having seen the data.

Bayesian inference for a normal subpopulation

- Prior density: $\theta \sim N(\mu, \tau^2)$
- Sampling density: $y_1, \dots, y_n | \theta \sim N(\theta, \sigma^2)$.

Bayes rule: $\theta | y_1, \dots, y_n$ is normal, with

$$E[\theta | y_1, \dots, y_n] = \frac{\tau^2}{\sigma^2/n + \tau^2} \bar{y} + \frac{\sigma^2/n}{\sigma^2/n + \tau^2} \mu$$

$$\text{Var}[\theta | y_1, \dots, y_n] = 1/(n/\sigma^2 + 1/\tau^2)$$

Bayes estimator: Let $\hat{\theta} = E[\theta | y_1, \dots, y_n]$. Then

$$E[(\hat{\theta} - \theta)^2 | y_1, \dots, y_n] \leq E[(\tilde{\theta} - \theta)^2 | y_1, \dots, y_n]$$

for any estimator $\tilde{\theta}$.

Bayes interpretation

A Bayesian interpretation of $\hat{\theta}$:

- θ_j is some fixed quantity for group j ;
- $\theta_j \sim N(\mu, \tau^2)$ describes prior info about θ_j ;
- $\theta_j \sim N(\hat{\theta}_j, 1/(n_j/\sigma^2 + 1/\tau^2))$ describes posterior info about θ_j ;
- $\hat{\theta}_j$ is “where you think θ_j is”.

Practical considerations:

- μ, τ^2, σ^2 ;
- estimate these parameters with a “fully Bayesian procedures”, or
- use plug-in estimates (Empirical Bayes), obtained from data (ANOVA, lme4).

Again, the estimator is the same but the justification can be different.

Summary

$\hat{\theta} = (\hat{\theta}_1, \dots, \hat{\theta}_m)$ may be evaluated

- with composite risk or group-level risk
- pre-experimentally or post-experimentally

Pre-experimentally

- $\hat{\theta}$ has lower composite risk than $\bar{y}$.
- $\hat{\theta}_j$ has lower risk than $\bar{y}_j$ for most j , but higher risk for extreme j .

Post-experimentally

- $\hat{\theta}_j$ minimizes posterior risk - it is where you think θ_j is, if you believe the model.